

Exceptional symmetric domains

Guy Roos

ABSTRACT. We give the presentation of exceptional bounded symmetric domains using the Albert algebra and exceptional Jordan triple systems.

The first chapter is devoted to Cayley-Graves algebras, the second to exceptional Jordan triple systems. In the third chapter, we give a geometric description of the two exceptional bounded symmetric domains, their boundaries and their compactifications.

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E-mail address: guy.roos@normalesup.org .

Introduction

The classification of irreducible bounded symmetric complex domains is well-known. They fall into four infinite series — the “classical domains” — which can be defined as matrix spaces, using ordinary matrix operations and classical linear groups, and two “exceptional” domains, of respective complex dimension 16 and 27, which have no matrix description (i.e., no description in a matrix space involving the usual matrix operations).

The main purpose of these notes is to present an explicit algebraic and geometric description of the two exceptional domains, which can no longer be considered as “unknown”, as well as some tools on them.

Analysis and geometry of classical domains have been extensively studied, following the pioneer work of Hua Luokeng [3], which consists of a case-by-case study of the four classical series. A general theory for all bounded complex domains also exists, using either semi-simple Lie groups (see [1], [2]) or Jordan triple systems (see [5], [7]). The study of one particular classical series still provides a good insight for conjecturing properties valid for all bounded symmetric domains.

The explicit description of the exceptional domains, which was not known at the time of Hua’s book, has been available for at least 30 years. The description involves 3×3 matrices with entries in the Cayley–Graves algebra $\mathbb{O}_{\mathbb{C}}$ of complex octonions. As this algebra is *non-associative*, these matrices do not carry the usual interpretation of linear algebra theory and they do not build an associative matrix algebra for the usual matrix operations. However, the space $\mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$ of such matrices which are Hermitian with respect to Cayley conjugation can be endowed with the structure of a *Jordan algebra*, using a product which generalizes in a natural way the symmetrized product

$$x \circ y = \frac{1}{2}(xy + yx) \quad (0.1)$$

of ordinary square matrices. This algebra is known as the *Albert algebra* or *exceptional Jordan algebra*. It is the natural place to describe the exceptional symmetric domain of dimension 27. The second exceptional symmetric domain (of complex dimension 16) lives in the space $\mathcal{M}_{2,1}(\mathbb{O}_{\mathbb{C}})$ of 2×1 matrices with octonion entries. This space has some analogy with the space $\mathcal{M}_{p,q}(\mathbb{C})$ of ordinary rectangular matrices, endowed with the *Jordan triple product*

$$\{xyz\} = xy^*z + zy^*x, \quad (0.2)$$

where y^* denotes the Hermitian adjoint (transposed conjugate) of y . The space $\mathcal{M}_{2,1}(\mathbb{O}_{\mathbb{C}})$ also carries the structure of a *Jordan triple system*, which allows an algebraico-geometric description of the exceptional domain of dimension 16.

The Jordan algebra $\mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$ and the Jordan triple system $\mathcal{M}_{2,1}(\mathbb{O}_{\mathbb{C}})$ are *exceptional* not only because they are not part of an infinite series, but more fundamentally because their algebraic products *cannot* be related to some associative product by formulas like (0.1) or (0.2). But the explicit description of their algebraic structure, combined with the general theory of Jordan triple systems and bounded symmetric domains, provides easy access to the geometry and analysis on the two exceptional symmetric domains. After this preliminary work, it appears that exceptional domains are as easy (or not worse) to handle than classical ones. It also appears that these two domains are as representative as classical ones for exhibiting phenomena which lead to conjectures for all symmetric domains.

The first chapter of these notes is devoted to Cayley–Graves algebras, the second to exceptional Jordan triple systems. In the third chapter, we give a geometric description of the two exceptional bounded symmetric domains, their boundaries and their compactifications.

1. Cayley algebras

We denote by k the field \mathbb{R} of real numbers or the field \mathbb{C} of complex numbers. A k -algebra is a k -vector space A , endowed with a k -bilinear *product*

$$(x, y) \mapsto xy \\ A \times A \rightarrow A.$$

This product is not assumed to be commutative nor associative. But we shall assume it has a unit element $e \neq 0$; this unit element is also denoted by 1.

1.1. Composition algebras.

DEFINITION 1. A composition algebra (or Hurwitz algebra) over k is a pair (A, n) , where A is a k -algebra and n a non-singular quadratic form on A , which is multiplicative in the sense that

$$n(ab) = n(a)n(b) \quad (a, b \in A). \quad (1.1)$$

The form n is called the norm of the composition algebra, and $n(a)$ is called the norm of a .

It is clear that $n(e) = 1$. We will identify k and ke using $\lambda \mapsto \lambda e$. The elements of ke are called the *scalars* of A . For each $\lambda \in k$, we have $n(\lambda e) = \lambda^2$.

If $A = k$, there is a unique composition algebra structure over the k -vector space k , given by $n(\lambda) = \lambda^2$. This justifies the above identification $\lambda \mapsto \lambda e$ in a general composition algebra.

Denote by A° the opposite algebra of A (i.e., the same vector space with the opposite product $x \cdot y = yx$); clearly (A°, n) is also a composition algebra, which is called the *opposite composition algebra* to (A, n) .

In a composition algebra (A, n) , we denote by $(:)$ the bilinear form associated to n :

$$(a : b) = n(a + b) - n(a) - n(b) \quad (a, b \in A). \quad (1.2)$$

Note there is no $\frac{1}{2}$ factor in this definition, which implies $(a : a) = 2n(a)$. Then the relation (1.1) can also be written

$$2(ab : ab) = (a : a)(b : b).$$

Polarizing this identity with respect to the variable b yields

$$(ac : ad) = n(a)(c : d); \quad (1.3)$$

polarizing with respect to the variable a yields in the same way

$$(ac : bc) = (a : b)n(c). \quad (1.4)$$

Polarizing again this last identity with respect to c , we obtain

$$(ac : bd) + (ad : bc) = (a : b)(c : d). \quad (1.5)$$

Specializing this identity to $b \leftarrow 1$, $c \leftarrow a$, and using (1.3), we obtain for all $d \in A$

$$(a^2 : d) + n(a)(d : 1) = (a : 1)(a : d),$$

which is equivalent to

$$(a^2 - (a : 1)a + n(a)1 : d) = 0.$$

As n is assumed to be non-singular, this implies

$$a^2 - (a : 1)a + n(a)1 = 0.$$

Define the *trace* $t(a)$ in a composition algebra by

$$t(a) = (a : 1). \quad (1.6)$$

We have proved that each element a in a composition algebra satisfies the equation of degree 2

$$a^2 - t(a)a + n(a)1 = 0. \quad (1.7)$$

1.2. Cayley conjugation. Let (A, n) be a composition algebra. The (*Cayley*) *conjugate* of an element $a \in A$ is defined by

$$\tilde{a} = (a : e)e - a. \quad (1.8)$$

The *Cayley conjugation* $a \mapsto \tilde{a}$ is the orthogonal symmetry (with respect to the quadratic form n) which has ke as its fixed point set. Therefore it is involutive and isometric:

$$(\tilde{a})^\sim = a, \quad n(\tilde{a}) = n(a). \quad (1.9)$$

The defining relation (1.8) can also be written $a + \tilde{a} = t(a)$; the identity (1.7) is then equivalent to $-a\tilde{a} + n(a) = 0$. So norm, trace and conjugation are related by the relations

$$a + \tilde{a} = t(a), \quad a\tilde{a} = \tilde{a}a = n(a) = n(\tilde{a}). \quad (1.10)$$

We also have, polarizing $n(\tilde{a}) = n(a)$,

$$(a : b) = (\tilde{a} : \tilde{b}). \quad (1.11)$$

The relation (1.5) with $b \leftarrow 1$ gives

$$(ac : d) + (ad : c) = ((a : 1)c : d);$$

as $(a : 1) = a + \tilde{a}$, we obtain $(ad : c) = (\tilde{a}c : d)$. The symmetric relation $(da : c) = (c\tilde{a} : d)$ is proved in the same way. These two identities can be better written as follows:

$$(ax : y) = (x : \tilde{a}y), \quad (1.12)$$

$$(xa : y) = (x : y\tilde{a}). \quad (1.13)$$

Using these identities, we have $(ab : 1) = (a : \tilde{b}) = (ba : 1)$, that is

$$t(ab) = t(ba); \quad (1.14)$$

we will say that the trace is “commutative” (with respect to the product). Using again the identities (1.12)-(1.13), we have

$$t((ab)c) = (ab : \tilde{c}) = (a : \tilde{c}\tilde{b}) = (ca : \tilde{b}) = t((ca)b),$$

which means that $t((ab)c)$ is invariant under *even* permutations of (a, b, c) . Using this fact and (1.14), we get

$$t((ab)c) = t((ca)b) = t((bc)a) = t(a(bc)),$$

that is, the trace is “associative” in the sense that

$$t((ab)c) = t(a(bc)). \quad (1.15)$$

From (1.12)–(1.13), we also have

$$(\tilde{a}\tilde{b} : c) = (ab : \tilde{c}) = (ca : \tilde{b}) = (c : \tilde{b}\tilde{a})$$

for all $c \in A$, which implies

$$(ab)^\sim = \tilde{b}\tilde{a}. \quad (1.16)$$

This means that the Cayley conjugation $a \mapsto \tilde{a}$ is an isomorphism from the composition algebra (A, n) onto the opposite algebra (A°, n) .

Using (1.5) and (1.12), we get

$$(a : b)(c : d) = (\tilde{b}(ac) : d) + (\tilde{a}(bc) : d)$$

for all $d \in A$, which implies, as $(:)$ is non-singular,

$$(a : b)c = \tilde{b}(ac) + \tilde{a}(bc). \quad (1.17)$$

In the same way (or using the isomorphism with the opposite algebra), we have

$$(a : b)c = (ca)\tilde{b} + (cb)\tilde{a}. \quad (1.18)$$

Specializing these two relations to the case $a = b$, we get

$$n(a)c = \tilde{a}(ac) = (ca)\tilde{a}. \quad (1.19)$$

The first equality can also be written $(\tilde{a}a)c = \tilde{a}(ac)$; using the fact that $a + \tilde{a}$ is scalar, this implies $a^2c = a(ac)$. One proves in the same way the identity $(ca)a = ca^2$. So we have proved that the following identities:

$$a^2c = a(ac), \quad (ca)a = ca^2 \quad (1.20)$$

are verified in a composition algebra.

DEFINITION 2. *An algebra which satisfies the identities (1.20) is called an alternative algebra.*

The property of being alternative will be referred to as *alternativity*.

1.3. Alternative algebras. Let A be a k -algebra. The *commutator* $[x, y]$ and the *associator* $[x, y, z]$ are respectively defined by

$$\begin{aligned} [x, y] &= xy - yx, \\ [x, y, z] &= x(yz) - (xy)z. \end{aligned}$$

These two multilinear maps provide an easy way for stating commutativity or associativity of the algebra A : the algebra A is commutative if and only if the commutator is identically 0, it is associative if the associator map is 0. The associator is also useful for characterizing alternativity.

PROPOSITION 1.1. *A k -algebra A is alternative if and only if the associator map $(x, y, z) \mapsto [x, y, z]$ is alternating.*

In fact, the identities (1.20) can be written equivalently

$$[a, a, c] = 0, \quad [c, a, a] = 0.$$

They are obviously verified if the associator is alternating.

Conversely, let A be alternative; then $[x, y, z]$ is 0 for $x = y$ or $y = z$. This means that $[x, y, z]$ is alternating with respect to (x, y) and with respect to (y, z) . As the transpositions (12) and (23) generate the symmetric group \mathfrak{S}_3 , it follows that the associator is a trilinear alternating map.

As a consequence, in an alternative algebra, we have

$$[a, b, a] = 0,$$

which can also be written

$$a(ba) = (ab)a. \quad (1.21)$$

An algebra satisfying (1.21) is called *flexible*. In such an algebra, we will simply write aba for $a(ba) = (ab)a$. In a composition algebra (A, n) , as $a + \tilde{a}$ is a scalar (a multiple of e), the identity (1.21) is equivalent to

$$\tilde{a}(ba) = (\tilde{a}b)a. \quad (1.22)$$

In an alternative algebra, we have the important *Moufang identities*:

THEOREM 1.2 (Ruth Moufang). *In an alternative algebra, the following identities are true:*

$$a(x(ay)) = (axa)y, \quad (1.23)$$

$$((xa)ya) = x(aya), \quad (1.24)$$

$$(ax)(ya) = a(xy)a. \quad (1.25)$$

They are called respectively the *left*, *right* and *central* Moufang identity.

PROOF. From the definitions, we get

$$a(x(ay)) - (axa)y = [a, x, ay] + [ax, a, y];$$

the right hand side is symmetric in (x, y) , so it is enough to check that it vanishes for $x = y$. We have $[a, x, ax] = [ax, a, x]$ by Proposition 1.1; repeatedly using (1.20), we obtain

$$[a, ax, x] = a((ax)x) - (a(ax))x = a^2x^2 - a^2x^2 = 0.$$

This proves the left identity (1.23). The right identity (1.24) is proved in the same way; we also note that it is just the left identity in the opposite algebra, which is also alternative.

We also get from the definitions and from alternativity

$$\begin{aligned} (ax)(ya) - a(xy)a &= -[a, x, ya] + a[x, y, a] \\ &= [a, ya, x] + a[y, a, x] \\ &= a(y(ax)) - ((aya)x). \end{aligned}$$

The last expression vanishes by (1.23), so the central identity (1.25) is proved. \square

The following proposition allows us to characterize composition algebras among alternative algebras:

PROPOSITION 1.3. *Let A be an algebra with unit element e . Assume there is an involutive anti-automorphism $a \mapsto \tilde{a}$ of A (with $\tilde{\tilde{e}} = e$) such that $a + \tilde{a}$ and $a\tilde{a}$ are scalars (multiples of e) for all $a \in A$. Define $n : A \rightarrow k$ by*

$$n(a) = a\tilde{a}.$$

Then (A, n) is a composition algebra if and only if A is alternative and n is non-singular. In this case, the Cayley conjugation in (A, n) is $a \mapsto \tilde{a}$.

PROOF. Let $a, b \in A$; then $a + \tilde{a} = \alpha$, $b + \tilde{b} = \beta$ with $\alpha, \beta \in ke$. We have then, using alternativity and the central Moufang identity,

$$\begin{aligned} n(ab) &= (ab) (ab)^\sim = (ab)(\tilde{b}\tilde{a}) = (ab)((\beta - b)(\alpha - a)) \\ &= ab\beta\alpha - ab^2\alpha - aba\beta + ab^2a \\ &= a(b(\beta - b))(\alpha - a) = a(b\tilde{b})\tilde{a} = n(b)a\tilde{a} \\ &= n(a)n(b). \end{aligned}$$

This shows that n is multiplicative; if n is non-singular, (A, n) is a composition algebra. The bilinear form associated to n is then $(a : b) = a\tilde{b} + b\tilde{a}$, which shows that the trace is $t(a) = (a : e) = a + \tilde{a}$ and that \tilde{a} is indeed the Cayley conjugate of a in (A, n) . \square

1.4. Cayley-Dickson extensions: analysis. We are going to describe the *Cayley-Dickson extension process*: start from the subalgebra $A_0 = ke$. This process allows one to construct successive subalgebras A_1, A_2, A_3 , each time doubling the dimension (as vector space) and terminates at most on the third step.

Let us first examine when a subalgebra B of a composition algebra (A, n) is itself a composition algebra.

PROPOSITION 1.4. *Let (A, n) and (B, n') be composition algebras with unit elements e, e' . If $f : B \rightarrow A$ is an algebra homomorphism (with $fe' = e$) and f is injective, then f is a (partial) isometry:*

$$n(fx) = n'(x) \quad (x \in B).$$

PROOF. Let $x \in B$ and let $y = fx$. We have $x^2 + t'(x)x + n'(x)e' = 0$, which gives

$$y^2 + t'(x)y + n'(x)e = 0;$$

comparing with $y^2 + t(y)y + n(y)e = 0$, we get

$$(t(y) - t'(x))y + (n(y) - n'(x))e = 0.$$

If (y, e) is free, then $n(y) = n'(x)$. If $y = \lambda e$, it follows from the injectivity of f that $x = \lambda e'$, and then again $n(y) = n'(x) = \lambda^2$. \square

Proposition 1.4 shows that if (B, n') is a composition subalgebra of (A, n) , the norm of B has to be the restriction of the norm of A . If (A, n) is a composition algebra and B is a subalgebra, then B is a composition subalgebra of (A, n) if and only if $n|_B$ is non-singular, that is, if $B \cap B^\perp = 0$.

PROPOSITION 1.5. *Let (A, n) be a composition algebra and let B a composition subalgebra. Assume $B \neq A$. Let $v \in B^\perp$ be non-isotropic: $n(v) = -\mu \neq 0$. Then*

1) the vector subspace vB is orthogonal to B and the map $\gamma_v : x \mapsto vx$ is an isomorphism from B onto vB ;

2) the subalgebra $C = \langle B, v \rangle$ generated by B and v is (as a vector space) $C = B \oplus vB$;

3) C is a composition subalgebra;

4) the product in $C = B \oplus vB$ is defined by

$$(a_1 + vb_1)(a_2 + vb_2) = a_1a_2 + \mu b_2\tilde{b}_1 + v(\tilde{a}_1b_2 + a_2b_1); \quad (1.26)$$

the norm and the Cayley conjugation are defined by

$$n(a + vb) = n(a) - \mu n(b), \quad (1.27)$$

$$(a + vb)^\sim = \tilde{a} - vb. \quad (1.28)$$

PROOF. As $v \perp B$, we have in particular $(v : 1) = 0$, which implies $\tilde{v} = -v$ and $v^2 = -n(v) = \mu$. For each $b \in B$, we have

$$0 = (b : v) = \tilde{b}v + \tilde{v}b = \tilde{b}v - vb,$$

which implies

$$vb = \tilde{b}v, \quad (vb)^\sim = -vb. \quad (1.29)$$

This proves (1.28).

Let $a, b \in B$; then $(a : vb) = (a\tilde{b} : v) = 0$. This shows $B \perp vB$. As $B \cap B^\perp = 0$, we have $B \cap vB = 0$. The relation $v(vx) = \mu x$ proves that $\gamma_v : x \mapsto vx$ is an isomorphism from B onto vB . The relation (1.27) directly follows from $B \perp vB$ and $n(v) = -\mu$. It shows that the restriction of n to $B \oplus vB$ is non-singular. It remains to prove the relations (1.26), which will imply that $C = \langle B, v \rangle = B \oplus vB$ and that C is a composition subalgebra.

Using (1.29), the central Moufang identity and alternativity, we get

$$(vb_1)(vb_2) = (vb_1)(\tilde{b}_2v) = v(b_1\tilde{b}_2)v = v^2b_2\tilde{b}_1 = \mu b_2\tilde{b}_1.$$

Using the left Moufang identity and (1.29), we have

$$v(a_1(vb_2)) = (va_1v)b = (v^2\tilde{a}_1)b_2 = \mu\tilde{a}_1b_2$$

and, multiplying by $\mu^{-1}v$,

$$a_1(vb_2) = v(\tilde{a}_1b_2).$$

Conjugating (after $b_2 \leftarrow b_1$ and $a_1 \leftarrow \tilde{a}_2$), we obtain $(\tilde{b}_1v)a_2 = (\tilde{b}_1\tilde{a}_2)v$ and, using again (1.29),

$$(vb_1)a_2 = v(a_2b_1).$$

□

1.5. Cayley-Dickson extensions: construction. Let (A, n) be a composition algebra. Let $A' = A \times A$. In view of Proposition 1.5, we consider on the vector space A' the product defined by

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 + \mu b_2\tilde{b}_1, \tilde{a}_1b_2 + a_2b_1)$$

and the quadratic form n' defined by

$$n'(a, b) = n(a) - \mu n(b);$$

we ask whether (A', n') is a composition algebra. In this case, it also follows from Proposition 1.5 that the conjugation in (A', n') will be given by

$$(a, b)^\sim = (\tilde{a}, -b).$$

With these definitions, A can be identified with the subalgebra $A \times 0$ of A' , by $a \mapsto (a, 0)$; the norm and conjugation in A' then extend those of A . If we set $v = (0, e)$, then $v(b, 0) = (0, b)$. So we can write $A' = A \oplus vA$ and the operation rules in A' are given by (1.26), (1.27), (1.28). It is easily seen that the conjugation in A' is an involutive antiautomorphism. Moreover, we have $(a, b) + (a, b)^\sim = t(a)(e, 0)$ and $(a, b)(a, b)^\sim = (a, b)(\tilde{a}, -b) = n'(a, b)(e, 0)$. Also, the definition of n' shows that it is non-singular if $\mu \neq 0$. So all conditions of Proposition 1.3 are fulfilled by A' with the product (1.26) and the conjugation (1.28), except alternativity. The answer to this last question is given by the following proposition.

PROPOSITION 1.6. *Let (A, n) be a composition algebra. For $\mu \neq 0$, denote by $A(\mu)$ the algebra*

$$A(\mu) = A \oplus vA$$

with the product

$$(a_1 + vb_1)(a_2 + vb_2) = a_1a_2 + \mu b_2\tilde{b}_1 + v(\tilde{a}_1b_2 + a_2b_1).$$

Then

- 1) $A(\mu)$ is commutative if and only if $A = ke$;
- 2) $A(\mu)$ is associative if and only if A is associative and commutative;
- 3) $A(\mu)$ is alternative if and only if A is associative.

PROOF. 1) The definition of the product in $A(\mu)$ implies $av = v\tilde{a}$ for all $a \in A$. If $A(\mu)$ is commutative, we have $va = v\tilde{a}$, which implies $a = \tilde{a}$ and $a \in ke$ for all $a \in A$. This shows $A = ke$.

Conversely, the algebra $k(\mu)$ with the product

$$(a_1 + vb_1)(a_2 + vb_2) = a_1a_2 + \mu b_1b_2 + v(a_1b_2 + a_2b_1)$$

is clearly commutative (and associative).

2) If $A(\mu)$ is associative, A is also associative. For $a, b \in A$, it is easily checked that

$$[a, b, v] = v[\tilde{a}, \tilde{b}];$$

so $A(\mu)$ associative implies A commutative.

Conversely, assume A is associative; then routine computations using the definitions show that, for $x, y, z \in A$, one has

$$\begin{aligned} [vx, y, z] &= v([y, z]x), & [x, vy, z] &= v([\tilde{x}, z]y), & [x, y, vz] &= v([\tilde{x}, \tilde{y}]z), \\ [x, vy, vz] &= \mu[x, z\tilde{y}], & [vx, y, vz] &= \mu[\tilde{y}, z\tilde{x}], & [vx, vy, z] &= \mu[z, y\tilde{x}], \\ [vx, vy, vz] &= \mu v([z\tilde{y}, x] + x[z, \tilde{y}]). \end{aligned}$$

If moreover A is commutative, we see that $A(\mu)$ is associative.

3) Let $x, y \in A(\mu)$. As $x + \tilde{x}$ is a scalar, we have

$$[x, x, y] = -[x, \tilde{x}, y]$$

and

$$[y, x, x]^\sim = -[y, x, \tilde{x}]^\sim = [x, \tilde{x}, y].$$

This shows that $A(\mu)$ is alternative if and only if $[x, \tilde{x}, y] = 0$ for all $x, y \in A(\mu)$. Now assume A is alternative and let $x = x_1 + vx_2$, $y = y_1 + y_2$; then

$$[x, \tilde{x}, y] = -\mu[x_1, y_2, \tilde{x}_2] + v[\tilde{x}_1, y_1, x_2],$$

which shows that $A(\mu)$ is alternative if and only if A is associative. \square

THEOREM 1.7. *A composition algebra is (as a vector space) of dimension 1, 2, 4 or 8.*

PROOF. Let A be a composition algebra over k . Let $A_0 = ke$. If $A \neq A_0$, there exists $v_1 \perp e$, with $n(v_1) = -\mu_1 \neq 0$; the composition subalgebra $A_1 = ke \oplus kv_1$ is commutative and associative. If $A \neq A_1$, there exists $v_2 \perp A_1$, with $n(v_2) = -\mu_2 \neq 0$; the composition subalgebra $A_2 = A_1 \oplus v_2 A_1$ is associative, but not commutative, of dimension 4. If $A \neq A_2$, there exists $v_3 \perp A_2$, with $n(v_3) = -\mu_3 \neq 0$; the composition subalgebra $A_3 = A_2 \oplus v_3 A_2$ is alternative (as A_2 is associative), but not associative (as A_2 is not commutative), of dimension 8. Then $A = A_3$, as A_3 is not associative. \square

1.6. Classification of composition algebras over \mathbb{R} or \mathbb{C} . We consider the composition algebras

$$k, \quad k(\mu_1), \quad k(\mu_1, \mu_2) = (k(\mu_1))(\mu_2), \quad k(\mu_1, \mu_2, \mu_3) = (k(\mu_1, \mu_2))(\mu_3),$$

for non-zero $\mu_1, \mu_2, \mu_3 \in k$. It follows from the proof of Theorem 1.7 that each composition algebra is isomorphic to one of these for a suitable choice of μ_1, μ_2, μ_3 . We want to make this statement more precise. First, we show that, if the norms of two composition algebras are linearly equivalent, these composition algebras are isomorphic.

PROPOSITION 1.8. *Let (A, n) and (A', n') be composition algebras. Then A and A' are isomorphic (as unital algebras) if and only there exists a linear isomorphism $f : A \rightarrow A'$ such that $n' \circ f = n$.*

PROOF. By Proposition 1.4, an isomorphism of composition algebras preserves norms.

Assume there exists a linear isomorphism $f : A \rightarrow A'$ such that $n' \circ f = n$. Let B, B' be proper composition subalgebras of A, A' respectively, such that there exists an algebra isomorphism $g : B \rightarrow B'$; then $n' \circ g = n|_B$ by Proposition 1.4. By Witt's theorem, g can be extended to a vector space isomorphism $h : A \rightarrow A'$ such that $n' \circ h = n$. Let $v \in B^\perp$ such that $n(v) = -\mu \neq 0$; take $v' = h(v)$, which implies $v' \in B'^\perp$ and $n'(v') = n(v) = -\mu$. Let

$$\widehat{g} : B \oplus vB \rightarrow B' \oplus v'B'$$

be defined by

$$\widehat{g}(a + vb) = g(a) + v'g(b).$$

By Proposition 1.5, \widehat{g} is then an algebra isomorphism between the composition subalgebras $B \oplus vB$ and $B' \oplus v'B'$. Starting from the trivial isomorphism $g_0 : ke \rightarrow ke'$ and iterating this process at most three times, we get an algebra isomorphism from A onto A' . \square

Assume that the ground field is $k = \mathbb{C}$. In each dimension, there is only one non-singular quadratic form, up to linear equivalence. So Proposition 1.8 implies:

THEOREM 1.9. *On $k = \mathbb{C}$, there exist, up to isomorphism, exactly four composition algebras A_j ($0 \leq j \leq 3$), of respective dimension 2^j .*

Assume now that the ground field is $k = \mathbb{R}$. In this case, non-singular quadratic forms are classified, up to linear equivalence, by their signature. The signature for $A_0^+ = \mathbb{R}$ is $(1, 0)$. Let us show that for other composition algebras over \mathbb{R} , the signature needs to be $(2a, 0)$ or (a, a) . If (B, n) is a composition algebra, we know that the norm of the Cayley-Dickson extension $B(\mu)$ is given by $n'(a + vb) = n(a) - \mu n(b)$. The signature of n' is

- $(2a, 0)$ if the signature of n is $(a, 0)$ and $\mu < 0$;
- (a, a) if the signature of n is $(a, 0)$ and $\mu > 0$;
- $(2b, 2b)$ if the signature of n is (b, b) .

The assumption on the signature can then be proved by induction.

THEOREM 1.10. *On $k = \mathbb{R}$, there exist, up to isomorphism, seven composition algebras:*

- the “compact” algebras A_j^+ ($0 \leq j \leq 3$) of dimension 2^j , with positive-definite norm;*
- the “split” algebras A_j^- ($1 \leq j \leq 3$) of dimension 2^j and signature $(2^{j-1}, 2^{j-1})$.*

When $k = \mathbb{C}$, a model for the composition algebra of dimension 4 is $A_2 \cong \mathcal{M}_{2,2}(\mathbb{C})$ (2×2 complex matrices), with the determinant as norm; a model for A_1 is the subalgebra of diagonal 2×2 complex matrices. The non-associative composition algebra A_3 is called the *complex Cayley algebra* or the *algebra of complex octonions*. It can be constructed, for example, as $A_2(-1)$; but this is in most cases irrelevant and it will be more important to know that this composition algebra of dimension 8 exists and is unique up to isomorphism. The algebra A_3 will be denoted by $\mathbb{O}_{\mathbb{C}}$.

In the case $k = \mathbb{R}$, models for the compact composition algebras of dimension 2 and 4 are respectively $A_1^+ \cong \mathbb{C}$ (with norm $n(z) = |z|^2$) and $A_2^+ = \mathbb{H}$ (the field of quaternions), which can be described as

$$\mathbb{H} = \left\{ q = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}; a, b \in \mathbb{C} \right\},$$

with norm $n(q) = a\bar{a} + b\bar{b}$. The compact non-associative real composition algebra A_3^+ is known as the *algebra of Cayley numbers*, the *algebra of octonions* or the *Cayley real division algebra*. It will be denoted by \mathbb{O} or $\mathbb{O}_{\mathbb{C}}$; it can be constructed as $\mathbb{H}(-1)$. Again the most important point is that \mathbb{O} is a real composition algebra of dimension 8 with positive norm, and is unique up to isomorphism.

The split composition algebras A_1^- and A_2^- are respectively isomorphic to the algebra of diagonal 2×2 real matrices and to the algebra $\mathcal{M}_{2,2}(\mathbb{R})$ of 2×2 real matrices, with the determinant as norm. The algebra A_3^- can be constructed as $\mathbb{R}(1, 1, 1)$; the signature of its norm is $(4, 4)$. It is denoted by \mathbb{O}_s and called the *split Cayley algebra*.

The real composition algebras can be *complexified* in a natural way. The complexification is then isomorphic to the complex composition algebra of the corresponding dimension.

2. Exceptional Jordan triple systems

2.1. The space $H_3(\mathbb{O})$. In this section, \mathbb{O} denotes a Cayley algebra over $k = \mathbb{R}$ or \mathbb{C} .

DEFINITION 3. *The space $H_3(\mathbb{O})$ is the k -vector space (with the natural operations) of 3×3 matrices with entries in \mathbb{O} , which are Hermitian with respect to the Cayley conjugation in \mathbb{O} .*

An element $a \in H_3(\mathbb{O})$ will be written

$$a = \begin{pmatrix} \alpha_1 & a_3 & \tilde{a}_2 \\ \tilde{a}_3 & \alpha_2 & a_1 \\ a_2 & \tilde{a}_1 & \alpha_3 \end{pmatrix}, \quad (2.1)$$

with $\alpha_1, \alpha_2, \alpha_3 \in k$ and $a_1, a_2, a_3 \in \mathbb{O}$. Instead of (2.1), we will also write

$$a = \sum_{j=1}^3 \alpha_j e_j + \sum_{j=1}^3 F_j(a_j), \quad (2.2)$$

with the obvious definitions for e_j and $F_j(a_j)$. The vector space $H_3(\mathbb{O})$ decomposes into the direct sum

$$H_3(\mathbb{O}) = ke_1 \oplus ke_2 \oplus ke_3 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3, \quad (2.3)$$

where $\mathcal{F}_j = \{F_j(a) \mid a \in \mathbb{O}\}$. The subspaces \mathcal{F}_j are 8-dimensional and

$$\dim_k H_3(\mathbb{O}) = 27.$$

On $H_3(\mathbb{O})$, define a bilinear form by

$$(a : b) = \sum_{j=1}^3 \alpha_j \beta_j + \sum_{j=1}^3 (a_j : b_j) \quad (2.4)$$

for

$$\begin{aligned} a &= \sum_{j=1}^3 \alpha_j e_j + \sum_{j=1}^3 F_j(a_j), \\ b &= \sum_{j=1}^3 \beta_j e_j + \sum_{j=1}^3 F_j(b_j); \end{aligned}$$

in (2.4), $(a_j : b_j)$ denotes the scalar product in \mathbb{O} . The form defined by (2.4) is clearly non-singular and the decomposition (2.3) is orthogonal with respect to it. We will refer to $(a : b)$ as the *scalar product* of a and b in $H_3(\mathbb{O})$.

DEFINITION 4. *The adjoint $a^\#$ of an element $a \in H_3(\mathbb{O})$, written in the form (2.2), is defined by*

$$a^\# = \sum_i (\alpha_j \alpha_k - n(a_i)) e_i + \sum_i \tilde{F}_i(a_j a_k - \alpha_i \tilde{a}_i). \quad (2.5)$$

In (2.5) and below, \sum_i means $\sum_{i=1}^3$ and j, k are defined by (i, j, k) being an even permutation of $(1, 2, 3)$; $\tilde{F}_i(c)$ stands for $F_i(\tilde{c})$.

DEFINITION 5. *The symmetric bilinear map, associated to the quadratic map $a \mapsto a^\#$, is called the Freudenthal product. The Freudenthal product of $a, b \in H_3(\mathbb{O})$ is denoted $a \times b$ and is defined by*

$$a \times b = (a + b)^\# - a^\# - b^\#, \quad a \times a = 2a^\#,$$

It follows directly from the definitions that

$$\begin{aligned} a \times b &= \sum_i (\alpha_j \beta_k + \alpha_k \beta_j - (a_i : b_i)) e_i \\ &\quad + \sum_i \tilde{F}_i(a_j b_k + b_j a_k - \alpha_i \tilde{b}_i - \beta_i \tilde{a}_i). \end{aligned} \quad (2.6)$$

The following multiplication rules hold:

$$\begin{aligned} e_i \times e_i &= 0, \quad e_i \times e_j = e_k, \\ e_i \times F_i(b) &= -F_i(b), \quad e_i \times F_j(b) = 0, \\ F_i(a) \times F_i(b) &= -(a : b)e_i, \quad F_i(a) \times F_j(b) = \widetilde{F}_k(ab), \end{aligned} \quad (2.7)$$

where (i, j, k) always stands for an even permutation of $(1, 2, 3)$.

PROPOSITION 2.1.

$$(a \times b : c) = (a : b \times c) \quad (a, b, c \in H_3(\mathbb{O})). \quad (2.8)$$

PROOF. For

$$\begin{aligned} a &= \sum_{j=1}^3 \alpha_j e_j + \sum_{j=1}^3 F_j(a_j), \quad b = \sum_{j=1}^3 \beta_j e_j + \sum_{j=1}^3 F_j(b_j), \\ c &= \sum_{j=1}^3 \gamma_j e_j + \sum_{j=1}^3 F_j(c_j), \end{aligned}$$

by applying the definitions we obtain

$$\begin{aligned} (a \times b : c) &= \sum_i (\alpha_j \beta_k + \alpha_k \beta_j - (a_i : b_i)) \gamma_i \\ &\quad + \sum_i (a_j b_k + b_j a_k - \alpha_i \tilde{b}_i - \beta_i \tilde{a}_i : \tilde{c}_i) \\ &= \sum_{(i,j,k) \in \mathfrak{S}_3} \alpha_i \beta_j \gamma_k + \sum_{(i,j,k) \in \mathfrak{S}_3} t(a_i, b_j, c_k) \\ &\quad - \sum_i ((a_i : b_i) \gamma_i + (b_i : c_i) \alpha_i + (c_i : a_i) \beta_i). \end{aligned}$$

(Recall that $t(x, y, z) = t((xy)z) = (xy : \tilde{z})$ for $x, y, z \in \mathbb{O}$). This shows that $(a \times b : c)$ is symmetric with respect to (a, b, c) . \square

DEFINITION 6. Let T denote the trilinear symmetric form on $H_3(\mathbb{O})$ defined by

$$T(a, b, c) = (a \times b : c).$$

The determinant in $H_3(\mathbb{O})$ is the associated polynomial of degree 3, defined by

$$\det a = \frac{1}{3!} T(a, a, a) = \frac{1}{3} (a^\# : a). \quad (2.9)$$

From the expression of $(a \times b : c)$, we deduce

$$\det a = \alpha_1 \alpha_2 \alpha_3 - \sum_i \alpha_i n(a_i) + a_1(a_2 a_3) + (\tilde{a}_3 \tilde{a}_2) \tilde{a}_1. \quad (2.10)$$

This relation may also be taken as a definition of $\det a$. It is an extension of the classical ‘‘Sarrus’ rule’’ for 3×3 matrices, but with suitable parentheses in products like $a_1(a_2 a_3)$, due to the non-associativity of the Cayley algebra.

PROPOSITION 2.2.

$$(a^\#)^\# = (\det a)a. \quad (2.11)$$

PROOF. Let

$$\begin{aligned} a &= \sum_{j=1}^3 \alpha_j e_j + \sum_{j=1}^3 F_j(a_j), \quad a^\# = \sum_{j=1}^3 \beta_j e_j + \sum_{j=1}^3 F_j(b_j), \\ (a^\#)^\# &= \sum_{j=1}^3 \gamma_j e_j + \sum_{j=1}^3 F_j(c_j). \end{aligned}$$

From the definition (2.5) and the properties of Cayley algebras, we get

$$\begin{aligned} \gamma_i &= \beta_j \beta_k - n(b_i) \\ &= (\alpha_k \alpha_i - n(a_j)) (\alpha_i \alpha_j - n(a_k)) - n(a_j a_k - \alpha_i \tilde{a}_i) \\ &= \alpha_i^2 \alpha_j \alpha_k - \alpha_i \alpha_j n(a_j) - \alpha_i \alpha_k n(a_k) + n(a_j) n(a_k) \\ &\quad - n(a_j a_k) - \alpha_i^2 n(a_j) + \alpha_i (a_j a_k : \tilde{a}_i) \\ &= \alpha_i \det a \end{aligned}$$

(using namely $n(a_j)n(a_k) = n(a_j a_k)$) and

$$\begin{aligned} c_i &= \tilde{b}_k \tilde{b}_j - \beta_i b_i \\ &= (a_i a_j - \alpha_k \tilde{a}_k) (a_k a_i - \alpha_j \tilde{a}_j) - (\alpha_j \alpha_k - n(a_i)) (\tilde{a}_k \tilde{a}_j - \alpha_i a_i) \\ &= (a_i a_j) (a_k a_i) - \alpha_k n(a_k) a_i - \alpha_j n(a_j) a_i + n(a_i) \tilde{a}_k \tilde{a}_j \\ &\quad + \alpha_i \alpha_j \alpha_k a_i - \alpha_i n(a_i) a_i \\ &= (\det a) a_i \end{aligned}$$

(here we used the central Moufang identity

$$(a_i a_j)(a_k a_i) = (a_i(a_j a_k)) a_i$$

and $n(a_i) \tilde{a}_k \tilde{a}_j = ((\tilde{a}_k \tilde{a}_j) \tilde{a}_i) a_i$. \square

PROPOSITION 2.3. *The following identities hold in $H_3(\mathbb{O})$:*

$$\det(a^\#) = (\det a)^2; \quad (2.12)$$

$$d(\det a).b = (a^\# : b); \quad (2.13)$$

$$a^\# \times (a \times b) = (\det a)b + (a^\# : b)a; \quad (2.14)$$

$$(a \times b : a^\# \times c) = (\det a)(b : c) + (a^\# : b)(a : c); \quad (2.15)$$

$$a \times (a^\# \times c) = (\det a)c + (a : c)a^\#; \quad (2.16)$$

$$\begin{aligned} (a \times b) \times (a \times c) + a^\# \times (b \times c) &= \\ &= (a^\# : b)c + (a^\# : c)b + T(a, b, c)a; \end{aligned} \quad (2.17)$$

$$\begin{aligned} a \times ((a \times b) \times c) + b \times (a^\# \times c) &= \\ &= (a^\# : b)c + (b : c)a^\# + (a : c)a \times b; \end{aligned} \quad (2.18)$$

$$a^\# \times b^\# + (a \times b)^\# = (a^\# : b)b + (b^\# : a)a; \quad (2.19)$$

$$(a \times b^\# : a^\# \times b) = 3 \det a \det b + (a : b)(a^\# : b^\#). \quad (2.20)$$

PROOF. We have

$$\det(a^\#) = \frac{1}{3} (a^\# : (a^\#)^\#) = \frac{1}{3} (a^\# : \det a \ a) = (\det a)^2.$$

By differentiating the relation $\det a = \frac{1}{6} T(a, a, a)$, we get

$$d(\det a).b = \frac{1}{2} T(a, a, b) = (a^\# : b).$$

We obtain (2.14) and (2.17) by successive differentiations of (2.11). The identity (2.15) is obtained from (2.14) by taking the scalar product with c and using (2.8). Using (2.8) again and the fact that $(:)$ is non-singular, we deduce (2.16) from (2.15). The relation (2.18) is obtained by differentiating (2.16). The relation (2.19) is (2.17) with $b = c$, and the identity (2.20) is (2.15) with $b = c^\#$. \square

2.2. The Hermitian Jordan triple system $H_3(\mathbb{O})$. For the definition and general properties of Jordan triple systems, we refer the reader to [4], [5], [7].

Let \mathbb{O}_c be the compact Cayley algebra over \mathbb{R} , with norm n and Euclidean associated scalar product $(:)$. We consider the complex Cayley algebra \mathbb{O} as the complexification of \mathbb{O}_c : $\mathbb{O} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}_c$; the product, the Cayley conjugation and the norm on \mathbb{O} are defined as the natural extensions of those on \mathbb{O}_c : $(\alpha \otimes a)(\beta \otimes b) = \alpha\beta \otimes ab$, $\widetilde{(\alpha \otimes a)} = \alpha \otimes \tilde{a}$ and $n(\alpha \otimes a) = \alpha^2 n(a)$ ($\alpha, \beta \in \mathbb{C}$, $a, b \in \mathbb{O}_c$). In addition, the algebra \mathbb{O} has a complex conjugation with respect to its “real form” \mathbb{O}_c , defined by

$$\overline{(\alpha \otimes a)} = \bar{\alpha} \otimes a \quad (\alpha \in \mathbb{C}, a \in \mathbb{O}_c);$$

this conjugation is antilinear and satisfies $\overline{ab} = \bar{a}\bar{b}$, in contrast with the Cayley conjugation which is complex linear and satisfies $\tilde{ab} = \tilde{b}\tilde{a}$.

The space $H_3(\mathbb{O})$, with the operations defined in the previous section, is then the complexification of the space $H_3(\mathbb{O}_c)$ with the same operations. If

$$a = \sum_{j=1}^3 \alpha_j e_j + \sum_{j=1}^3 F_j(a_j) \in H_3(\mathbb{O}),$$

its complex conjugate with respect to $H_3(\mathbb{O}_c)$ is defined by

$$\bar{a} = \sum_{j=1}^3 \bar{\alpha}_j e_j + \sum_{j=1}^3 F_j(\bar{a}_j).$$

Clearly we have

$$\bar{a}^\# = \overline{a^\#}, \quad \bar{a} \times \bar{b} = \overline{a \times b}, \quad \det \bar{a} = \overline{\det a}.$$

On \mathbb{O} and $H_3(\mathbb{O})$, we define the Hermitian scalar product

$$(a | b) = (a : \bar{b}).$$

DEFINITION 7. *The triple product $\{xyz\}$ on $H_3(\mathbb{O})$, and the related operators Q and D , are defined by*

$$Q(x)y = (x | y)x - x^\# \times \bar{y}, \tag{2.21}$$

$$D(x, y)z = \{xyz\} = (x | y)z + (z | y)x - (x \times z) \times \bar{y}. \tag{2.22}$$

PROPOSITION 2.4. *With this triple product, $H_3(\mathbb{O})$ is a Hermitian Jordan triple system.*

PROOF. The triple product defined by (2.22) is clearly \mathbb{C} -bilinear symmetric in (x, z) and antilinear in y . We are going to prove that it satisfies the defining identities (J1) and (J2) of a Jordan triple system.

Let us prove

$$D(x, y)Q(x) = Q(x)D(y, x). \quad (\text{J1})$$

We have

$$\begin{aligned} D(x, y)Q(x)u &= (x | y)Q(x)u + (Q(x)u | y)x - (x \times Q(x)u) \times \bar{y} \\ &= (x | y)((x | u)x - x^\# \times \bar{u}) + ((x | u)x - x^\# \times \bar{u} | y)x \\ &\quad - (x \times ((x | u)x - x^\# \times \bar{u})) \times \bar{y} \\ &= 2(x | y)(x | u)x - (x | y)x^\# \times \bar{u} - (x^\# | y \times u)x \\ &\quad - 2(x | u)x^\# \times \bar{y} + (x \times (x^\# \times \bar{u})) \times \bar{y}; \end{aligned}$$

using (2.16): $x \times (x^\# \times \bar{u}) = (\det x)\bar{u} + (x | u)x^\#$, we get

$$\begin{aligned} D(x, y)Q(x)u &= 2(x | y)(x | u)x - (x | y)x^\# \times \bar{u} - (x | u)x^\# \times \bar{y} \\ &\quad - (x^\# | y \times u)x + (\det x)\bar{y} \times \bar{u}. \end{aligned}$$

In the same way,

$$\begin{aligned} Q(x)D(y, x)u &= (x | D(y, x)u)x - x^\# \times \overline{D(y, x)u} \\ &= (x | (y | x)u + (u | x)y - (y \times u) \times \bar{x})x \\ &\quad - x^\# \times \left((x | y)\bar{u} + (x | u)\bar{y} - \overline{(y \times u) \times x} \right) \\ &= 2(x | y)(x | u)x - 2(x^\# | y \times u)x - (x | y)x^\# \times \bar{u} \\ &\quad - (x | u)x^\# \times \bar{y} + x^\# \times \left(x \times \overline{(y \times u)} \right); \end{aligned}$$

using (2.14): $x^\# \times \left(x \times \overline{(y \times u)} \right) = \det x \overline{(y \times u)} + (x^\# | y \times u)x$, we get

$$Q(x)D(y, x)u = D(x, y)Q(x)u.$$

This proves (J1).

Let us now prove

$$D(Q(x)y, y) = D(x, Q(y)x). \quad (\text{J2})$$

We have

$$\begin{aligned} D(Q(x)y, y)z &= (Q(x)y | y)z + (z | y)Q(x)y - (Q(x)y \times z) \times \bar{y} \\ &= ((x | y)x - x^\# \times \bar{y} | y)z + (z | y)((x | y)x - x^\# \times \bar{y}) \\ &\quad - ((x | y)x - x^\# \times \bar{y}) \times z \times \bar{y} \\ &= (x | y)^2 z - 2(x^\# | y^\#)z + (z | y)(x | y)x - (z | y)x^\# \times \bar{y} \\ &\quad - (x | y)(x \times z) \times \bar{y} + ((x^\# \times \bar{y}) \times z) \times \bar{y} \end{aligned}$$

and

$$\begin{aligned} D(x, Q(y)x)z &= (x | Q(y)x)z + (z | Q(y)x)x - (x \times z) \times \overline{Q(y)x} \\ &= (x | (y | x)y - y^\# \times \bar{x})z + (z | (y | x)y - y^\# \times \bar{x})x \\ &\quad - (x \times z) \times ((x | y)\bar{y} - \bar{y}^\# \times x) \end{aligned}$$

$$\begin{aligned}
&= (x \mid y)^2 z - 2(x^\# \mid y^\#)z + (z \mid y)(x \mid y)x - (z \times x \mid y^\#)x \\
&\quad - (x \mid y)(x \times z) \times \bar{y} + (x \times z) \times (\bar{y}^\# \times x).
\end{aligned}$$

Applying (2.17) gives

$$(x \times z) \times (x \times \bar{y}^\#) + x^\# \times (\bar{y}^\# \times z) = (x^\# \mid y^\#)z + (x^\# \mid z)\bar{y}^\# + (z \times x \mid y^\#)x;$$

applying (2.18) yields

$$((x^\# \times \bar{y}) \times z) \times \bar{y} + x^\# \times (\bar{y}^\# \times z) = (x^\# \mid y^\#)z + (x^\# \mid z)\bar{y}^\# + (z \mid y)x^\# \times \bar{y}.$$

Comparing these two last identities gives

$$(x \times z) \times (x \times \bar{y}^\#) - (z \times x \mid y^\#)x = ((x^\# \times \bar{y}) \times z) \times \bar{y} - (z \mid y)x^\# \times \bar{y},$$

which implies $D(Q(x)y, y)z = D(x, Q(y)x)z$ and proves (J2). \square

The space $H_3(\mathbb{O})$ endowed with the triple product defined by (2.22) will be referred to as the *Hermitian Jordan triple system* $H_3(\mathbb{O})$, or the *Hermitian JTS of type VI*, or the *exceptional Hermitian JTS of dimension 27*.

The real subspace $H_3(\mathbb{O}_c)$, which is clearly a real Jordan triple subsystem, and a “real form” of $H_3(\mathbb{O})$ in the sense that the triple product in $H_3(\mathbb{O})$ is obtained from the product in $H_3(\mathbb{O}_c)$ by suitable “complexification”, will be called the *Euclidean JTS* $H_3(\mathbb{O}_c)$, or the *Euclidean JTS of type VI*, or the *exceptional compact JTS of dimension 27*.

2.3. The minimal polynomial of $H_3(\mathbb{O})$. In this section, we compute the *generic minimal polynomial* and the *rank* of the Jordan triple system $H_3(\mathbb{O})$ (see [7] for the general theory of these notions in a JTS). Recall that the powers $x^{(k,y)}$ in a Hermitian Jordan triple system V are defined for $x, y \in V$ and $k \in \mathbb{N}$, $k > 0$ by

$$\begin{aligned}
x^{(1,y)} &= x, \\
x^{(k+1,y)} &= \frac{1}{2}D(x, y)x^{(k,y)},
\end{aligned}$$

and the *odd powers* $x^{(2k+1)}$ of $x \in V$, for $k \in \mathbb{N}$, by

$$x^{(2k+1)} = x^{(k+1,x)}.$$

A *tripotent element* in $H_3(\mathbb{O})$ is an element x such that $x^{(3)} = x$.

LEMMA 2.5. *Let $x, y \in H_3(\mathbb{O})$. Then*

$$x^{(2,y)} = \frac{1}{2}D(x, y)x = (x \mid y)x - x^\# \times \bar{y},$$

$$\frac{1}{2}D(x, y)(x^\# \times \bar{y}) = (x^\# \mid y^\#)x - \det x \bar{y}^\#, \quad (2.23)$$

$$\frac{1}{2}D(x, y)\bar{y}^\# = \det \bar{y} x. \quad (2.24)$$

The subspace $\sum_1^\infty \mathbb{C}x^{(k,y)}$ is contained in the subspace generated by $(x, x^\# \times \bar{y}, \bar{y}^\#)$; the flat subspace generated by x :

$$<< x >> = \sum_0^\infty \mathbb{C}x^{(2k+1)}$$

is contained in the subspace generated by $(x, x^\# \times \bar{x}, \bar{x}^\#)$.

PROOF. The relation for $x^{(2,y)}$ is nothing but the defining relation. From (2.22) and (2.16), we have

$$\begin{aligned} D(x, y) (x^\# \times \bar{y}) &= (x | y) x^\# \times \bar{y} + (x^\# \times \bar{y} | y) x - (x \times (x^\# \times \bar{y})) \times \bar{y} \\ &= (x | y) x^\# \times \bar{y} + 2(x^\# | y^\#) x - ((\det x) \bar{y} + (x | y) x^\#) \times \bar{y} \\ &= 2(x^\# | y^\#) x - 2 \det x \bar{y}^\#, \end{aligned}$$

that is, (2.23). Using (2.16) again, we get

$$\begin{aligned} D(x, y) \bar{y}^\# &= (x | y) \bar{y}^\# + (\bar{y}^\# | y) x - (x \times \bar{y}^\#) \times \bar{y} \\ &= (x | y) \bar{y}^\# + 3 \det \bar{y} x - \det \bar{y} x - (x | y) \bar{y}^\# \\ &= 2 \det \bar{y} x, \end{aligned}$$

that is, (2.24). \square

PROPOSITION 2.6. *The generic minimal polynomial of the Jordan triple system $H_3(\mathbb{O})$ is*

$$m(T, x, y) = T^3 - (x | y) T^2 + (x^\# | y^\#) T - \det x \det \bar{y}; \quad (2.25)$$

the rank of $H_3(\mathbb{O})$ is 3.

PROOF. The lemma shows that the JTS $H_3(\mathbb{O})$ has rank ≤ 3 . It is now a matter of elementary algebra to compute a linear relation between x , $x^{(2,y)}$, $x^{(3,y)}$, $x^{(4,y)}$. From (2.23), (2.24), we deduce

$$\begin{aligned} x^{(3,y)} &= \frac{1}{2} D(x, y) x^{(2,y)} = (x | y) x^{(2,y)} - (x^\# | y^\#) x + \det x \bar{y}^\#, \\ x^{(4,y)} &= \frac{1}{2} D(x, y) x^{(3,y)} = (x | y) x^{(3,y)} - (x^\# | y^\#) x^{(2,y)} + \det x \det \bar{y} x. \end{aligned}$$

This shows that for all $x, y \in V = H_3(\mathbb{O})$, the minimal polynomial of x in $V^{(y)}$ divides

$$T^3 - (x | y) T^2 + (x^\# | y^\#) T - \det x \det \bar{y}, \quad (2.26)$$

and so does the generic minimal polynomial. In order to prove that this is actually the generic minimal polynomial, we take

$$x = y = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$$

with $\alpha_1 > \alpha_2 > \alpha_3 > 0$. As it is easily checked, (e_1, e_2, e_3) is a set of *orthogonal tripotents* (i.e. $D(e_i, e_i) e_j = 2\delta_{ij} e_j$) and the minimal polynomial of x in $V^{(x)}$ is $(T - \alpha_1^2)(T - \alpha_2^2)(T - \alpha_3^2)$. This shows that the generic minimal polynomial has to be of degree 3 and is equal to (2.26). \square

2.4. Positivity; tripotents. The next proposition implies that $H_3(\mathbb{O})$ is a *positive* Hermitian Jordan triple system.

PROPOSITION 2.7. *Let $x \in H_3(\mathbb{O})$, $x \neq 0$. Then $x^{(3)} = \lambda x$ if and only if one of the following occurs:*

- (1) $(x | x) = \lambda$, $x^\# = 0$;
- (2) $(x | x) = 2\lambda$, $(x^\# | x^\#) = \lambda^2$, $\det x = 0$;
- (3) $(x | x) = 3\lambda$, $(x^\# | x^\#) = 3\lambda^2$, $|\det x|^2 = \lambda^3$.

PROOF. By definition $x^{(3)} = (x \mid x)x - x^\# \times \bar{x}$, so the relation $x^{(3)} = \lambda x$ is equivalent to

$$((x \mid x) - \lambda)x = x^\# \times \bar{x}. \quad (2.27)$$

As $\sum_1^\infty \mathbb{C}x^{(2k-1)}$ is contained in the subspace generated by $(x, x^\# \times \bar{x}, \bar{x}^\#)$, the relation (2.27) holds if and only if both sides have the same Hermitian products with $x, x^\# \times \bar{x}, \bar{x}^\#$. This provides the conditions

$$((x \mid x) - \lambda)(x \mid x) = 2(x^\# \mid x^\#), \quad (2.28)$$

$$\begin{aligned} 2((x \mid x) - \lambda)(x^\# \mid x^\#) &= (x^\# \times \bar{x} \mid x^\# \times \bar{x}) \\ &= 3|\det x|^2 + (x \mid x)(x^\# \mid x^\#), \end{aligned}$$

(using (2.20)), that is,

$$((x \mid x) - 2\lambda)(x^\# \mid x^\#) = 3|\det x|^2 \quad (2.29)$$

and finally, using

$$2(x^\# \times \bar{x} \mid x^\#) = 4(\bar{x} \mid (x^\#)^\#) = 4(x \mid x) \det x,$$

we get

$$3((x \mid x) - \lambda) \det x = 2(x \mid x) \det x,$$

that is,

$$((x \mid x) - 3\lambda) \det x = 0. \quad (2.30)$$

If $x \neq 0, x^\# = 0$, then (2.29), (2.30) are satisfied and (2.28) is equivalent to $(x \mid x) = \lambda$.

If $x^\# \neq 0$ but $\det x = 0$, then (2.30) is satisfied. Condition (2.29) is equivalent to $(x \mid x) = 2\lambda$ and (2.28) is then equivalent to $(x^\# \mid x^\#) = \lambda^2$.

If $\det x \neq 0$, (2.30) is equivalent to $(x \mid x) = 3\lambda$; (2.28) provides $(x^\# \mid x^\#) = 3\lambda^2$ and (2.29) gives $|\det x|^2 = \lambda^3$. \square

As an immediate consequence, we have

PROPOSITION 2.8. *The set \mathcal{E} of tripotents of $H_3(\mathbb{O})$ is the disjoint union $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$, where $\mathcal{E}_0 = \{0\}$,*

$$\mathcal{E}_1 = \{x \mid (x \mid x) = 1, x^\# = 0\}, \quad (2.31)$$

$$\mathcal{E}_2 = \{x \mid (x \mid x) = 2, (x^\# \mid x^\#) = 1, \det x = 0\}, \quad (2.32)$$

$$\mathcal{E}_3 = \left\{x \mid (x \mid x) = 3, (x^\# \mid x^\#) = 3, |\det x|^2 = 1\right\}. \quad (2.33)$$

LEMMA 2.9. *Let $x, y \in H_3(\mathbb{O})$ be two orthogonal tripotents. Then $(x \mid y) = 0$.*

PROOF. Let x, y be two tripotents. They are orthogonal if and only if $D(x, y) = 0$.

From (2.24): $D(x, y)\bar{y}^\# = 2\det \bar{y} x$, we deduce that $x = 0$ or $\det y = 0$. So if $y \in \mathcal{E}_3$, then $x = 0$. Let $\det y = 0$; from (2.23):

$$\frac{1}{2}D(x, y)(x^\# \times \bar{y}) = (x^\# \mid y^\#)x - \det x \bar{y}^\#,$$

we then deduce

$$\frac{1}{2}(D(x, y)(x^\# \times \bar{y}) \mid y) = (x^\# \mid y^\#)(x \mid y) - 3\det x \det \bar{y},$$

which implies $(x^\# \mid y^\#)(x \mid y) = 0$. Also,

$$0 = x^{(2,y)} = \frac{1}{2}D(x, y)x = (x \mid y)x - x^\# \times \overline{y},$$

which implies $(x \mid y)x = x^\# \times \overline{y}$ and $(x \mid y)^2 = 2(x^\# \mid y^\#)$; hence

$$(x \mid y)^3 = 2(x^\# \mid y^\#)(x \mid y) = 0.$$

□

It follows from Lemma 2.9 that if $x \in \mathcal{E}_i$ and $y \in \mathcal{E}_j$ are orthogonal tripotents, then their sum $x + y$, which is also a tripotent, belongs to \mathcal{E}_{i+j} . In particular, elements of \mathcal{E}_1 are minimal tripotents and elements of \mathcal{E}_3 are maximal tripotents. The elements of \mathcal{E}_3 have the following simple characterization:

PROPOSITION 2.10. *An element $x \in H_3(\mathbb{O})$ is in \mathcal{E}_3 if and only if $x \neq 0$ and*

$$x = \det x \overline{x^\#}. \quad (2.34)$$

PROOF. If $x \in \mathcal{E}_3$, then $2x = x^\# \times \overline{x}$, which implies

$$4x^\# = x \times (x^\# \times \overline{x}) = \det x \overline{x} + (x \mid x)x^\#.$$

Hence $x^\# = \det x \overline{x}$ (as $(x \mid x) = 3$) and $\det \overline{x} x^\# = \overline{x}$ (as $|\det x|^2 = 1$).

Conversely, let $x \neq 0$, $x = \det x \overline{x^\#}$; this implies $\det x \neq 0$. Then $x^\# \times \overline{x} = x^\# \times (\det \overline{x} x^\#) = 2|\det x|^2 x$. This means that $x^{(3)} = \lambda x$, with $\lambda = (x \mid x) - 2|\det x|^2$. By Proposition 2.7, we have $|\det x|^2 = \lambda^3$ and $(x \mid x) = 3\lambda$, which implies $(x \mid x) = 3|\det x|^2$ and $\lambda = \lambda^3 > 0$; hence $\lambda = 1$ and $x \in \mathcal{E}_3$. □

For a tripotent x in a positive Hermitian JTS V , the operator $D(x, x)$ is self-adjoint and its eigenvalues are in $\{0, 1, 2\}$. The *Peirce decomposition* relative to x is

$$V = V_2(x) \oplus V_1(x) \oplus V_0(x),$$

where the *Peirce subspaces* $V_j(x)$ are the eigenspaces of $D(x, x)$:

$$V_j(x) = \{y \in V \mid D(x, x)y = jy\}, \quad j = 0, 1, 2.$$

PROPOSITION 2.11. *For $x \in \mathcal{E}_3$, the Peirce subspaces are $V_0(x) = V_1(x) = 0$, $V_2(x) = V = H_3(\mathbb{O})$.*

PROOF. It suffices to prove that for each $y \in H_3(\mathbb{O})$, one has $D(x, x)y = 2y$. As $D(x, x)x = 2x$, it is enough to prove this if $(x \mid y) = 0$. If $(x \mid y) = 0$, we have then, using (2.34) and (2.14),

$$\begin{aligned} D(x, x)y &= (x \mid x)y - (x \times y) \times \overline{x} \\ &= 3y - \det \overline{x} (x \times y) \times x^\# \\ &= 3y - \det \overline{x} (\det x y + (x^\# : y)x) = 2y, \end{aligned}$$

as $|\det x|^2 = 1$ and, by (2.34), $\det \overline{x} (x^\# : y) = (y \mid x) = 0$. □

Proposition 2.6 shows that a maximal flat subspace has dimension 3. From Proposition 2.8, we see that e_1, e_2, e_3 belong to \mathcal{E}_1 and are therefore minimal tripotents. From the definition

$$D(e_1, e_1)z = z + (z \mid e_1)e_1 - e_1 \times (e_1 \times z)$$

and from the relations (2.7), it is easily checked that

$$V_0(e_1) = \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathcal{F}_1, \quad (2.35)$$

$$V_1(e_1) = \mathcal{F}_2 \oplus \mathcal{F}_3, \quad (2.36)$$

$$V_2(e_1) = \mathbb{C}e_1. \quad (2.37)$$

Similar results hold for the Peirce decomposition with respect to e_2 and e_3 . As e_2 and e_3 belong to $V_0(e_1)$, they are orthogonal to e_1 ; also, e_2 is orthogonal to e_3 . So (e_1, e_2, e_3) is a frame for the Jordan triple system $H_3(\mathbb{O})$ and $\mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$ is a maximal flat subspace. It is also easily checked that the simultaneous Peirce decomposition with respect to the frame (e_1, e_2, e_3) is

$$H_3(\mathbb{O}) = \bigoplus_{1 \leq i \leq j \leq 3} V_{ij},$$

with $V_{ii} = \mathbb{C}e_i$, $V_{ij} = \mathcal{F}_k$.

THEOREM 2.12 (Freudenthal's theorem). *Let $x \in H_3(\mathbb{O})$. Then there exists $k \in \text{Aut } H_3(\mathbb{O})$ such that*

$$kx = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \quad (\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}).$$

Actually, $H_3(\mathbb{O})$ is a positive Jordan triple system; then for each x there exists an automorphism k such that kx belongs to the maximal flat subspace $\mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$. The theorem may also be proved directly in this special case, following the lines of the general theory.

THEOREM 2.13. *The Hermitian Jordan triple system $H_3(\mathbb{O})$ is simple and of tube type. Its numerical invariants are*

$$a = 8, \quad b = 0, \quad r = 3, \quad g = 18.$$

To show that $H_3(\mathbb{O})$ is simple, it is enough to find a frame such that all V_{ij} ($1 \leq i < j \leq 3$) are non-zero; this occurs with the frame (e_1, e_2, e_3) . We then have $a = \dim \mathcal{F}_i = 8$, $b = \dim V_{0i} = 0$, $g = 2 + a(r - 1) = 18$.

COROLLARY 2.14. *In the Jordan triple system $H_3(\mathbb{O})$, we have*

$$\text{Tr } D(x, y) = 18(x \mid y), \quad (2.38)$$

$$\text{Det } B(x, y) = (1 - (x \mid y) + (x^\# \mid y^\#) - \det x \det \overline{y})^{18}, \quad (2.39)$$

where Tr and Det denote the trace and determinant of \mathbb{C} -linear operators in $H_3(\mathbb{O})$.

2.5. The exceptional Jordan triple system of dimension 16. We consider the subsystem of $H_3(\mathbb{O})$

$$V_1(e_1) = \mathcal{F}_2 \oplus \mathcal{F}_3,$$

which is then a positive Hermitian Jordan triple system of dimension 16. Let us denote the space $V_1(e_1)$ by W . For $x = F_2(x_2) + F_3(x_3) \in W$, we have, according to (2.5),

$$x^\# = -n(x_2)e_2 - n(x_3)e_3 + \widetilde{F}_1(x_2x_3) \in V_0(e_1)$$

and $\det x = 0$. The structure of Jordan triple system in W is defined, for $x = F_2(x_2) + F_3(x_3)$, $y = F_2(y_2) + F_3(y_3)$, by

$$Q(x)y = (x \mid y)x - x^\# \times \overline{y}$$

$$\begin{aligned}
&= ((x_2 | y_2) + (x_3 | y_3))x - n(x_2)F_2(\overline{y_2}) - n(x_3)F_3(\overline{y_3}) \\
&\quad - \widetilde{F_3}((\widetilde{x_3}\widetilde{x_2})\overline{y_2}) - \widetilde{F_2}(\overline{y_3}(\widetilde{x_3}\widetilde{x_2})) \\
&= F_2\left((x_2 | y_2)x_2 + (x_3 | y_3)x_2 - n(x_2)\overline{y_2} - (x_2x_3)\widetilde{\overline{y_3}}\right) \\
&\quad + F_3\left((x_2 | y_2)x_3 + (x_3 | y_3)x_3 - n(x_3)\overline{y_3} - \widetilde{\overline{y_2}}(x_2x_3)\right).
\end{aligned}$$

Using identities in Cayley algebras, we get

$$Q(x)y = F_2\left(x_2\widetilde{\overline{y_2}}x_2 + (x_2\overline{y_3})\widetilde{x_3}\right) + F_3\left(\widetilde{x_2}(\overline{y_2}x_3) + x_3\widetilde{\overline{y_3}}x_3\right). \quad (2.40)$$

The triple product in W is then given by

$$\begin{aligned}
\{xyz\} &= F_2\left((x_2\widetilde{\overline{y_2}})z_2 + (z_2\widetilde{\overline{y_2}})x_2 + (x_2\overline{y_3})\widetilde{z_3} + (z_2\overline{y_3})\widetilde{x_3}\right) \\
&\quad + F_3\left(\widetilde{x_2}(\overline{y_2}z_3) + \widetilde{z_2}(\overline{y_2}x_3) + x_3(\widetilde{\overline{y_3}}z_3) + z_3(\widetilde{\overline{y_3}}x_3)\right). \quad (2.41)
\end{aligned}$$

PROPOSITION 2.15. *The generic minimal polynomial of W is*

$$m_W(T; x, y) = T^2 - (x | y)T + (x^\# | y^\#).$$

For $x, y \in W$, the subspace $\sum_1^\infty \mathbb{C}x^{(k,y)}$ is contained in $\mathbb{C}x + \mathbb{C}x^\# \times \overline{y}$; the flat subspace generated by x :

$$<< x >> = \sum_0^\infty \mathbb{C}x^{(2k+1)}$$

is contained in the subspace generated by $(x, x^\# \times \overline{x})$.

PROOF. Let $x, y \in W$; then, by (2.23) and $\det x = 0$, we have

$$\begin{aligned}
x^{(2,y)} &= \frac{1}{2}D(x, y)x = (x | y)x - x^\# \times \overline{y}, \\
\frac{1}{2}D(x, y)(x^\# \times \overline{y}) &= (x^\# | y^\#)x.
\end{aligned}$$

This shows $\sum_1^\infty \mathbb{C}x^{(k,y)} \subset \mathbb{C}x + \mathbb{C}x^\# \times \overline{y}$. Moreover, these relations imply

$$x^{(3,y)} = (x | y)x^{(2,y)} - (x^\# | y^\#)x,$$

which shows that the generic minimal polynomial $m(T, x, y) = m_W(T; x, y)$ divides $T^2 - (x | y)T + (x^\# | y^\#)$. To prove equality, it will be enough to prove that the rank of W is 2, that is, to find $x \in W$ such that x and $x^\# \times \overline{x}$ are \mathbb{R} -linearly independent. For this, take $x = F_2(b)$, with $b \in \mathbb{O}$; then $x^\# = -n(b)e_2$ and $x^\# \times \overline{x} = n(b)F_2(\overline{b})$. So it suffices to choose $b \in \mathbb{O}$ such that $n(b) = 1$ and b, \overline{b} linearly independent. This proves $m(T, x, y) = T^2 - (x | y)T + (x^\# | y^\#)$. \square

A tripotent in W is also a tripotent in $V = H_3(\mathbb{O})$; as $\det x = 0$ for each $x \in W$, $\mathcal{E}_3 \cap W = \emptyset$, the set of tripotents of W is $\mathcal{E}' = \mathcal{E}'_0 \cup \mathcal{E}'_1 \cup \mathcal{E}'_2$ with $\mathcal{E}'_j = \mathcal{E}_j \cap W$. Also, two orthogonal tripotents $x, y \in W$ are orthogonal in V and hence verify $(x | y) = 0$; it follows that elements of \mathcal{E}'_1 are minimal tripotents and elements of \mathcal{E}'_2 are maximal tripotents.

Minimal tripotents $F_2(\beta) + F_3(\gamma)$ are characterized by

$$n(\beta) = n(\gamma) = 0, \quad \beta\gamma = 0, \quad (\beta | \beta) + (\gamma | \gamma) = 1. \quad (2.42)$$

An example of minimal tripotent is then given by $u = F_2(\beta)$, with β satisfying

$$(\beta | \beta) = 1, \quad n(\beta) = 0; \quad (2.43)$$

these relations are equivalent to

$$\beta = b_1 + ib_2, \quad b_1, b_2 \in \mathbb{O}_c, \quad n(b_1) = n(b_2) = \frac{1}{4}, \quad (b_1 : b_2) = 0. \quad (2.44)$$

LEMMA 2.16. *Let $\beta \in \mathbb{O}$ such that $(\beta | \beta) = 1$ and $n(\beta) = 0$. Then for each $x \in \mathbb{O}$,*

$$x = \tilde{\beta}(\overline{\beta}x) + \tilde{\tilde{\beta}}(\beta x). \quad (2.45)$$

If $L(\beta)$ denotes the left multiplication by β in \mathbb{O} : $L(\beta)x = \beta x$, the following direct sum decomposition holds:

$$\mathbb{O} = \ker L(\beta) \oplus \ker L(\tilde{\beta}); \quad (2.46)$$

moreover,

$$\ker L(\beta) = \text{Im } L(\tilde{\beta})$$

and

$$\dim_{\mathbb{C}} \ker L(\beta) = 4.$$

PROOF. Polarizing the identity $n(y)x = \tilde{y}(yx)$ in \mathbb{O} , we obtain

$$(y : z)x = \tilde{y}(zx) + \tilde{z}(yx)$$

and (2.45) follows by $y \leftarrow \beta, z \leftarrow \overline{\beta}$. Let $x_1 = \tilde{\beta}(\overline{\beta}x)$ and $x_2 = \tilde{\tilde{\beta}}(\beta x)$; then $x_1 \in \ker L(\beta)$ and $x_2 \in \ker L(\tilde{\beta})$. If $x \in \ker L(\beta) \cap \ker L(\tilde{\beta})$, it follows from (2.45) that $x = 0$; this proves (2.46).

Clearly $\text{Im } L(\tilde{\beta}) \subset \ker L(\beta)$, by $\beta(\tilde{\beta}x) = n(\beta)x = 0$. Assume $\beta x = 0$; by (2.45), we have $x = \tilde{\beta}(\overline{\beta}x)$, that is $x \in \text{Im } L(\tilde{\beta})$. So $\ker L(\beta) = \text{Im } L(\tilde{\beta})$.

As $x \mapsto \overline{x}$ is a (real) automorphism of \mathbb{O} , the spaces $\ker L(\beta)$ and $\ker L(\tilde{\beta})$ have the same real dimension, hence also the same complex dimension. This implies $\dim_{\mathbb{C}} \ker L(\beta) = 4$. \square

We are now able to compute the Peirce decomposition with respect to the minimal tripotent $u = F_2(\beta)$.

LEMMA 2.17. *Let $\beta \in \mathbb{O}$ such that $(\beta | \beta) = 1$ and $n(\beta) = 0$. The spaces of the Peirce decomposition of W with respect to $u = F_2(\beta)$ are*

$$W_0(u) = \mathbb{C}\overline{u} \oplus F_3(\ker L(\tilde{\beta})), \quad (2.47)$$

$$W_1(u) = F_2(\langle \beta, \overline{\beta} \rangle^\perp) \oplus F_3(\ker L(\beta)), \quad (2.48)$$

$$W_2(u) = \mathbb{C}u. \quad (2.49)$$

Here $\langle \beta, \overline{\beta} \rangle^\perp$ stands for the orthogonal subspace of \mathbb{O} , with respect to the Hermitian product $(|)$, of the 2-dimensional subspace $\mathbb{C}\beta \oplus \mathbb{C}\overline{\beta}$. Note that the conditions on β mean that $(\beta, \overline{\beta})$ is orthonormal.

PROOF. For $x = F_2(x_2) + F_3(x_3)$, we have

$$\begin{aligned} D(u, u)x &= (u | u)x + (x | u)u - (u \times x) \times \overline{u} \\ &= F_2(x_2) + F_3(x_3) + (x_2 | \beta)F_2(\beta) \\ &\quad - (F_2(\beta) \times F_2(x_2)) \times F_2(\overline{\beta}) - (F_2(\beta) \times F_3(x_3)) \times F_2(\overline{\beta}) \\ &= F_2(x_2) + F_3(x_3) + (x_2 | \beta)F_2(\beta) + (x_2 : \beta)F_2(\overline{\beta}) - F_3(\tilde{\beta}(\beta x_3)); \end{aligned}$$

finally

$$\begin{aligned} D(u, u)x &= F_2(x_2 + (x_2 \mid \beta)\beta - (x_2 : \beta)\bar{\beta}) + F_3(x_3 - \widetilde{\beta}(\beta x_3)) \\ &= F_2(x_2 + (x_2 \mid \beta)\beta - (x_2 : \beta)\bar{\beta}) + F_3(\widetilde{\beta}(\bar{\beta}x_3)). \end{aligned}$$

From these two expressions of $D(u, u)x$, it is easily seen that

- $D(u, u)x = 0$ if $x_2 \in \mathbb{C}\bar{\beta}$ and $\bar{\beta}x_3 = 0$;
- $D(u, u)x = x$ if $(x_2 \mid \beta) = (x_2 \mid \bar{\beta}) = 0$ and $\beta x_3 = 0$;
- $D(u, u)x = 2x$ if $x_2 \in \mathbb{C}\beta$ and $x_3 = 0$.

This provides the diagonalization of $D(u, u)$ with the indicated eigenspaces. \square

It follows easily from Lemma 2.17 that $v = F_2(\bar{\beta})$ is a minimal tripotent, orthogonal to u . The eigenspaces of $D(v, v)$ are obtained from Lemma 2.17 with $\beta \leftarrow \bar{\beta}$. By comparing the two Peirce decompositions, we obtain

PROPOSITION 2.18. *The spaces of the simultaneous Peirce decomposition with respect to the frame $(u, v) = (F_2(\beta), F_2(\bar{\beta}))$ are*

$$W_{01} = F_3(\ker L(\beta)), \quad W_{02} = F_3(\ker L(\bar{\beta})), \quad (2.50)$$

$$W_{12} = F_2(\langle \beta, \bar{\beta} \rangle^\perp), \quad W_{11} = \mathbb{C}F_2(\beta), \quad W_{22} = \mathbb{C}F_2(\bar{\beta}). \quad (2.51)$$

PROPOSITION 2.19. *The triple system W is simple. Its numerical invariants are $a = 6$, $b = 4$, $r = 2$, $g = 12$. In W ,*

$$\text{Tr } D(x, y) = 12(x \mid y),$$

$$\text{Det } B(x, y) = (1 - (x \mid y) + (x^\# \mid y^\#))^{12}.$$

The set of tripotents of W is $\mathcal{E}' = \mathcal{E}'_0 \cup \mathcal{E}'_1 \cup \mathcal{E}'_2$, with $\mathcal{E}'_0 = \{0\}$,

$$\mathcal{E}'_1 = \{x \in W; (x \mid x) = 1, x^\# = 0\}, \quad (2.52)$$

$$\mathcal{E}'_2 = \{x \in W; (x \mid x) = 2, (x^\# \mid x^\#) = 1\}. \quad (2.53)$$

PROOF. The tripotents of W have already been described. From the previous proposition, we see that $\dim W_{12} = 6$ for the frame $(F_2(\beta), F_2(\bar{\beta}))$. This implies that W is simple, as it is positive as a subsystem of the positive Hermitian JTS $H_3(\mathbb{O})$. The numerical invariants are $r = 2$, $a = \dim W_{12} = 6$, $b = \dim W_{01} = 4$, $g = 2 + a(r - 1) + b = 12$. \square

As an example of maximal tripotent of W , we have $w = u + v = F_2(\beta + \bar{\beta}) = F_2(c)$, with $c \in \mathbb{O}_c$ and $n(c) = 1$. The Peirce spaces for w are $W_2(w) = \mathcal{F}_2$, $W_1(w) = \mathcal{F}_3$.

The simple positive JTS W is called the *exceptional JTS of dimension 16*.

Let us look at the Jordan structure of the Peirce subspaces with respect to the minimal tripotent u . The subspace $W_0(u)$ has rank 1 and is isomorphic to $I_{1,5}$. Consider

$$W' = W_1(u) = F_2(\langle \beta, \bar{\beta} \rangle^\perp) \oplus F_3(\ker L(\beta)).$$

Let $\gamma \in \langle \beta, \bar{\beta} \rangle^\perp$ such that $n(\gamma) = 0$ and $(\gamma \mid \gamma) = 1$. Then $u' = F_2(\gamma)$ and $v' = F_2(\bar{\gamma})$ are two orthogonal tripotents in W' and form a frame for W' . The spaces

of the total Peirce decomposition of W' with respect to this frame are obtained from the corresponding spaces in W by intersection with W' , which gives

$$W'_{01} = F_3(\ker L(\beta) \cap \ker L(\gamma)), \quad W'_{02} = F_3(\ker L(\beta) \cap \ker L(\bar{\gamma})),$$

$$W'_{12} = F_2(\langle \beta, \bar{\beta}, \gamma, \bar{\gamma} \rangle^\perp), \quad W'_{11} = \mathbb{C}F_2(\gamma), \quad W'_{22} = \mathbb{C}F_2(\bar{\gamma}).$$

Clearly $\dim W'_{12} = 4$, which implies that W' is simple; then $\dim W'_{01} = \dim W'_{02} = 2$.

The only simple positive Hermitian Jordan triple system with rank 2, dimension 10 and $a = 4$ is II_5 . This proves

PROPOSITION 2.20. *The Peirce subspace $W_1(u)$ of the exceptional JTS of type V with respect to a minimal tripotent u is of type II_5 .*

EXERCISE 2.1. Prove this proposition directly.

EXERCISE 2.2. Consider $v' = F_3(\gamma)$ with γ subject to the same conditions as β :

$$(\gamma \mid \gamma) = 1, \quad n(\gamma) = 0.$$

Then v' is another minimal tripotent.

- (1) Show that v' is orthogonal to $u = F_2(\beta)$ if and only if $\bar{\beta}\gamma = 0$.
- (2) Compute the simultaneous Peirce decomposition with respect to the frame $(F_2(\beta), \widetilde{F_3(\bar{\beta})})$.

EXERCISE 2.3. Let $\beta \in \mathbb{O}$ such that $(\beta \mid \beta) = 1$ and $n(\beta) = 0$.

- (1) Compute the Peirce decomposition of the JTS of type VI $H_3(\mathbb{O})$ with respect to the minimal tripotent $u = F_2(\beta)$.
- (2) Find a minimal tripotent f such that $(F_2(\beta), F_2(\bar{\beta}), f)$ is a frame of $H_3(\mathbb{O})$.

EXERCISE 2.4. Find a Jordan triple subsystem W' of $H_3(\mathbb{O})$, isomorphic to W , containing e_1 and e_2 . Compute the Peirce decomposition of W' with respect to (e_1, e_2) .

3. The exceptional symmetric domains

3.1. Description of exceptional symmetric domains. We apply the general results of [5]. As in the previous section, we denote by \mathbb{O} the algebra of complex octonions, by $V = H_3(\mathbb{O})$ the exceptional Jordan system with the Jordan triple structure defined by Definition 7, by $W = \mathcal{F}_2 \oplus \mathcal{F}_3$ the subsystem of dimension 16 studied in Subsection 2.5. Recall that these two complex Jordan triples are Hermitian positive and simple, with respective generic minimal polynomials

$$m_V(T, x, y) = T^3 - (x \mid y)T^2 + (x^\# \mid y^\#)T - \det x \det \bar{y}, \quad (3.1)$$

$$m_W(T, x, y) = T^2 - (x \mid y)T + (x^\# \mid y^\#). \quad (3.2)$$

For a Hermitian positive Jordan triple of rank r and generic minimal polynomial $m(T, x, y)$, the associated circled bounded symmetric domain is defined by the r inequalities

$$f_{k+1}(x, x) \equiv \frac{1}{k!} \left. \frac{d^k}{dT^k} m(T; x, x) \right|_{T=1} > 0 \quad (k = 0, \dots, r-1). \quad (3.3)$$

It follows that the symmetric domain $\Omega = \Omega_V$ associated to V (called the *exceptional symmetric domain of dimension 27*, or the *symmetric domain of type VI*) is the set of points in $H_3(\mathbb{O})$ which satisfy

$$f_1(x, x) \equiv 1 - (x | x) + (x^\# | x^\#) - |\det x|^2 > 0, \quad (3.4)$$

$$f_2(x, x) \equiv 3 - 2(x | x) + (x^\# | x^\#) > 0, \quad (3.5)$$

$$f_3(x, x) \equiv 3 - (x | x) > 0, \quad (3.6)$$

while the symmetric domain $\Omega' = \Omega_W$ associated to W (called the *exceptional symmetric domain of dimension 16*, or the *symmetric domain of type V*) is the set of points in $W = \mathcal{F}_2 \oplus \mathcal{F}_3$ which satisfy

$$g_1(x, x) \equiv 1 - (x | x) + (x^\# | x^\#) > 0, \quad (3.7)$$

$$g_2(x, x) \equiv 2 - (x | x) > 0. \quad (3.8)$$

3.2. Structure of the boundary.

3.2.1. *General results.* The inequalities (3.3) are equivalent to the fact that all roots of the polynomial $m(T; x, x)$ in T (which are always positive) are less than 1. The boundary of the symmetric domain Ω is the disjoint union of locally closed submanifolds $\partial_k \Omega$, which correspond to the case where 1 is a root of $m(T; x, x)$ with multiplicity k and the remaining roots are less than 1. We first recall general results, valid for each simple Hermitian positive JTS and the associated irreducible bounded symmetric domain (see for example [5], §§5-6). Then we apply these results to the case of the two exceptional symmetric domains.

The description of the boundary (Proposition 3.3) also involves the manifold of tripotents of the corresponding JTS, which is described in Proposition 3.2 below. The description of their tangent space needs a refinement of the Peirce decomposition $V = V_2(e) \oplus V_1(e) \oplus V_0(e)$ associated to a tripotent e .

PROPOSITION 3.1. *For a tripotent e in a Hermitian positive Jordan triple system V , the operator $Q(e)$ is zero on $V_1(e) \oplus V_0(e)$ and restricts to a \mathbb{C} -antilinear involution on $V_2(e)$.*

Denoting by $V_2^+(e)$, $V_2^-(e)$ the eigenspaces of $Q(e)$ for the eigenvalues $+1$, -1 :

$$V_2^+(e) = \{x \in V \mid Q(e)x = x\},$$

$$V_2^-(e) = \{x \in V \mid Q(e)x = -x\},$$

we have $V_2^-(e) = i V_2^+(e)$ and $V_2(e) = V_2^+(e) \oplus V_2^-(e)$. In any simple Hermitian positive Jordan triple system, one then has (see [5], Theorem 5.6):

PROPOSITION 3.2. *The set \mathcal{E}_k of tripotents of rank k is a compact connected submanifold, and the group K of automorphisms of the Jordan triple system acts transitively on \mathcal{E}_k . For $e \in \mathcal{E}_k$, the direction of the tangent space to \mathcal{E}_k at e is*

$$\overrightarrow{T_e \mathcal{E}_k} = i V_2^+(e) \oplus V_1(e).$$

The complex tangent space $H_e \mathcal{E}_k$ to \mathcal{E}_k at e has direction

$$\overrightarrow{H_e \mathcal{E}_k} = V_1(e).$$

The manifold \mathcal{E}_k is a Cauchy-Riemann manifold of CR type (s, t) and real dimension $2s + t$, with

$$s = \dim_{\mathbb{C}} V_1(e), \quad t = \dim_{\mathbb{R}} V_2^+(e) = \dim_{\mathbb{C}} V_2(e).$$

PROPOSITION 3.3. *Let V be a simple Hermitian positive JTS of rank r and let Ω be the associated irreducible symmetric domain. The boundary $\partial\Omega$ of the symmetric domain $\Omega = \Omega_V$ of type VI is the disjoint union*

$$\partial\Omega = \bigsqcup_{k=1}^r \partial_k\Omega \quad (3.9)$$

of locally closed manifolds

$$\partial_k\Omega = \{x \in V \mid f_j(x, x) = 0 \quad (1 \leq j \leq k), \quad f_m(x, x) > 0 \quad (m > k)\},$$

where the f_j 's are defined by (3.3). The “boundary part” $\partial_k\Omega$ contains the manifold \mathcal{E}_k of rank k tripotents in V . Each $\partial_{k+1}\Omega$ is contained in $\overline{\partial_k\Omega} \setminus \partial_k\Omega$.

For $e \in \mathcal{E}_k$, the normal direction at e to $\partial_k\Omega$ is $V_2^+(e)$ and the direction $\overrightarrow{T_e(\partial_k\Omega)}$ of the tangent space $T_e(\partial_k\Omega)$ is

$$\overrightarrow{T_e(\partial_k\Omega)} = iV_2^+(e) \oplus V_1(e) \oplus V_0(e).$$

The intersection of $\partial_k\Omega$ with the affine tangent space $e + \overrightarrow{T_e(\partial_k\Omega)}$ is

$$\partial_k\Omega \cap \left(e + \overrightarrow{T_e(\partial_k\Omega)}\right) = \partial_k\Omega \cap (e + V_0(e)) = e + \Omega(e),$$

where $\Omega(e)$ is the symmetric domain associated to the Jordan triple subsystem $V_0(e)$. The direction of the tangent space to $\partial_k\Omega$ is constant along $e + \Omega(e)$.

The boundary part $\partial_k\Omega$ is the disjoint union

$$\partial_k\Omega = \bigsqcup_{e \in \mathcal{E}_k} (e + \Omega(e)).$$

Let

$$p_k : \partial_k\Omega \rightarrow \mathcal{E}_k$$

be defined by $p_k(x) = e$ if $e \in \mathcal{E}_k$ and $x - e \in V_0(e)$. Then $(\partial_k\Omega, \mathcal{E}_k, p_k)$ is a locally trivial fiber bundle, isomorphic to $(\mathcal{X}_k, \mathcal{E}_k, q_k)$, where

$$\mathcal{X}_k = \{(e, y) \in \mathcal{E}_k \times V \mid y \in \Omega(e)\}$$

and q_k is the first projection. The boundary part $\partial_r\Omega$ is compact and equal to the manifold \mathcal{E}_r of maximal tripotents.

The boundary part $\partial_r\Omega = \mathcal{E}_r$ is actually the *Shilov boundary* of Ω , that is, the smallest set of points where the functions that are holomorphic on the domain and continuous up to the boundary take their maximum modulus values.

The (affine) submanifold $e + \overline{\Omega(e)}$ is called *affine component* of $\partial\Omega$ (through the minimal tripotent e). It can be shown that $e + \overline{\Omega(e)}$ is the maximal affine subset of $\partial\Omega$ containing e , which justifies its name. The decomposition (3.9) will be referred to as the *stratification* of the boundary. The submanifolds $\partial_k\Omega$ are called the *boundary parts* (or *strata*) of the boundary $\partial\Omega$. Clearly, one has $\partial_1\Omega = \partial\Omega$; the submanifold $\partial_1\Omega$ has real codimension 1 and is referred to as the *smooth part* of the boundary.

3.2.2. *The boundary of the exceptional domain of type VI.* Using Propositions 3.2, 3.3 and the results about tripotents from the previous section, we work out the details for the exceptional Jordan triple system $\mathcal{H}_3(\mathbb{O})$.

PROPOSITION 3.4. *Let $V = H_3(\mathbb{O})$ be the exceptional Jordan triple system of type VI. Then*

1. *The manifold \mathcal{E}_1 of minimal tripotents is*

$$\begin{aligned}\mathcal{E}_1 &= \{e \in V \mid \{e, e, e\} = 2e, (e \mid e) = 1\} \\ &= \{e \in V \mid e^\# = 0, (e \mid e) = 1\}.\end{aligned}$$

For $e \in \mathcal{E}_1$, we have $V_2(e) = \mathbb{C}e$, $\dim_{\mathbb{C}} V_1(e) = 16$. The manifold \mathcal{E}_1 has real dimension 33 and is a Cauchy-Riemann manifold of CR type $(16, 1)$.

2. *The manifold \mathcal{E}_2 of rank 2 tripotents is defined by*

$$\begin{aligned}\mathcal{E}_2 &= \{e \in V \mid \{e, e, e\} = 2e, (e \mid e) = 2\} \\ &= \{e \in V \mid \det e = 0, (e^\# \mid e^\#) = 1, (e \mid e) = 2\}.\end{aligned}$$

For $e \in \mathcal{E}_2$, we have $\dim V_2(e) = 10$ and $\dim V_1(e) = 16$. The manifold \mathcal{E}_2 has real dimension 42 and is a Cauchy-Riemann manifold of CR type $(16, 10)$.

3. *The manifold \mathcal{E}_3 of maximal tripotents is defined by*

$$\begin{aligned}\mathcal{E}_3 &= \{e \in V \mid \{e, e, e\} = 2e, (e \mid e) = 3\} \\ &= \left\{e \in V \mid |\det e|^2 = 1, (e^\# \mid e^\#) = 3, (e \mid e) = 3\right\}.\end{aligned}$$

The manifold \mathcal{E}_3 is totally real of real dimension 27.

PROOF. The characterization of the manifolds \mathcal{E}_k has been obtained in Proposition 2.8. To study the spaces $V_1(e)$ and $V_2(e)$, we use the fact that the group K of automorphisms of the Jordan triple system acts transitively on each \mathcal{E}_k and that $u(V_j(e)) = V_j(ue)$ for $u \in K$.

1. Consider the minimal tripotent $e = e_1$. Then

$$V_1(e_1) = \mathcal{F}_2 \oplus \mathcal{F}_3, \quad V_2(e_1) = \mathbb{C}e_1,$$

which yields for all $e \in \mathcal{E}_1$, $V_2(e) = \mathbb{C}e$, $\dim_{\mathbb{C}} V_1(e) = 16$.

2. Consider the tripotent of rank 2: $e = e_1 + e_2$. As e_1 and e_2 are orthogonal tripotents, we have

$$D(e, e) = D(e_1, e_1) + D(e_2, e_2).$$

From

$$\begin{aligned}V_0(e_1) &= \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathcal{F}_1, \\ V_1(e_1) &= \mathcal{F}_2 \oplus \mathcal{F}_3, \quad V_2(e_1) = \mathbb{C}e_1\end{aligned}$$

and the analogous statement for $V_j(e_2)$, we deduce

$$\begin{aligned}V_0(e) &= \mathbb{C}e_3, \quad V_1(e) = \mathcal{F}_1 \oplus \mathcal{F}_2, \\ V_2(e) &= \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathcal{F}_3.\end{aligned}$$

Hence we have $\dim V_2(e) = 10$ and $\dim V_1(e) = 16$ for all $e \in \mathcal{E}_2$.

3. Consider the maximal tripotent $e = e_1 + e_2 + e_3$. Then $V_2(e) = V$, $V_0(e) = V_1(e) = 0$. The tangent space direction to \mathcal{E}_3 at e is $V_2^+(e)$ and is totally real of real dimension 27. \square

We now work out the specific information for the application of Proposition 3.3 to the exceptional symmetric domain of dimension 27.

PROPOSITION 3.5. 1. *The smooth boundary part $\partial_1\Omega$ is a locally closed submanifold of real codimension 1, defined by*

$$\begin{aligned} f_1(x, x) &\equiv 1 - (x \mid x) + (x^\# \mid x^\#) - |\det x|^2 = 0, \\ f_2(x, x) &\equiv 3 - 2(x \mid x) + (x^\# \mid x^\#) > 0, \\ f_3(x, x) &\equiv 3 - (x \mid x) > 0. \end{aligned}$$

It contains the manifold

$$\mathcal{E}_1 = \{e \in H_3(\mathbb{O}); (e \mid e) = 1, e^\# = 0\}$$

of minimal tripotents of $H_3(\mathbb{O})$. For $e \in \mathcal{E}_1$, the normal direction at e to $\partial_1\Omega$ is e and the direction $\overrightarrow{T_e(\partial_1\Omega)}$ of the tangent space $T_e(\partial_1\Omega)$ is

$$\overrightarrow{T_e(\partial_1\Omega)} = i\mathbb{R}e \oplus V_1(e) \oplus V_0(e).$$

The Peirce subspaces $V_1(e)$ and $V_0(e)$ have respective complex dimensions 16 and 10. The Jordan triple subsystem $V_0(e)$ is isomorphic to the classical Hermitian JTS of type IV_{10} and the domain $\Omega(e) \subset V_0(e)$ is isomorphic to a Lie ball of dimension 10.

2. *The boundary part $\partial_2\Omega$ is a locally closed, Cauchy-Riemann submanifold of dimension 44 and CR type $(17, 10)$; it contains the manifold*

$$\mathcal{E}_2 = \{e \mid (e \mid e) = 2, (e^\# \mid e^\#) = 1, \det e = 0\}$$

of rank 2 tripotents in $H_3(\mathbb{O})$. For $e \in \mathcal{E}_2$, the normal direction $V_2^+(e)$ to $\partial_2\Omega$ at e has real dimension 10; the direction $\overrightarrow{T_e(\partial_2\Omega)}$ of the tangent space $T_e(\partial_2\Omega)$ is

$$\overrightarrow{T_e(\partial_2\Omega)} = iV_2^+(e) \oplus V_1(e) \oplus V_0(e),$$

where the Peirce subspaces $V_1(e)$ and $V_0(e)$ have respective complex dimensions 16 and 1. The intersection of $\partial_2\Omega$ with the affine tangent space $e + \overrightarrow{T_e(\partial_2\Omega)}$ is

$$\partial_2\Omega \cap \left(e + \overrightarrow{T_e(\partial_2\Omega)}\right) = \partial_2\Omega \cap (e + V_0(e)) = e + \Omega(e),$$

where $\Omega(e)$ is the unit disc of the one dimensional Jordan triple subsystem $V_0(e)$.

3. *The submanifold $\partial_3\Omega = \mathcal{E}_3$ is compact and totally real (of real dimension 27).*

PROOF. 1. Consider the minimal tripotent $e = e_1$. Then

$$V_2(e_1) = \mathbb{C}e_1, \quad V_1(e_1) = \mathcal{F}_2 \oplus \mathcal{F}_3, \quad V_0(e_1) = \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathcal{F}_1.$$

In the Jordan triple subsystem $V_0(e_1)$, a frame is given by (e_2, e_3) ; the spaces of the Peirce decomposition of $T = V_0(e_1)$ with respect to this frame are

$$\begin{aligned} T_2(e_2) &= \mathbb{C}e_2, \quad T_2(e_3) = \mathbb{C}e_3, \quad T_1(e_2) \cap T_1(e_3) = \mathcal{F}_1, \\ T_1(e_2) \cap T_0(e_3) &= T_0(e_2) \cap T_1(e_3) = 0. \end{aligned}$$

This shows that the Hermitian positive JTS is simple, with rank $r = 2$ and multiplicities $a = 8, b = 0$. The only possibility shown by the classification of Hermitian positive JTS is the type IV_{10} . The isomorphism of $V_0(e_1)$ with the standard JTS of type IV_{10} can also be checked directly (see Exercise 3.1).

2. Consider the rank 2 tripotent $e = e_1 + e_2$. Then

$$V_0(e) = \mathbb{C}e_3, \quad V_1(e) = \mathcal{F}_1 \oplus \mathcal{F}_2, \quad V_2(e) = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathcal{F}_3,$$

$\dim V_2(e) = 10$ and $\dim V_1(e) = 16$. The submanifold $\partial_2\Omega$ has normal direction $V_2^+(e)$, hence codimension 10 and dimension 44; the complex tangent direction to $\partial_2\Omega$ at e is $V_0(e) \oplus V_1(e) = \mathbb{C}e_3 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2$ and has dimension 17, which means $\partial_2\Omega$ has CR type $(17, 10)$. The one-dimensional subsystem $V_0(e) = \mathbb{C}e_3$ admits as tripotent e_3 , and we have $\Omega(e) = \Delta e_3$, where Δ is the unit disc of \mathbb{C} .

3. The equality $\partial_3\Omega = \mathcal{E}_3$ results from Proposition 3.3 and is also easily checked directly. The properties of \mathcal{E}_3 have been obtained in Proposition 3.4. \square

EXERCISE 3.1. Compute the Jordan triple product in $V_0(e_1)$ and show that the JTS $V_0(e_1)$ is isomorphic to the Hermitian positive JTS of type IV_{10} .

EXERCISE 3.2. 1. For a minimal tripotent $e = \mathcal{E}_1$, compute explicitly the Peirce subspaces $V_1(e)$ and $V_0(e)$ and their Jordan triple structure.

2. Compute the map

$$p_1 : \partial_1\Omega \rightarrow \mathcal{E}_1$$

defined by $p_1(x) = e$ if $x \in \mathcal{E}_1$ and $x - e \in V_0(e)$, using the operations in $\mathcal{H}_3(\mathbb{O})$.

EXERCISE 3.3. 1. For a rank 2 tripotent $e = \mathcal{E}_2$, compute explicitly the Peirce subspaces $V_1(e)$ and $V_0(e)$ and their Jordan triple structure.

2. Compute the map

$$p_2 : \partial_2\Omega \rightarrow \mathcal{E}_2$$

defined by $p_2(x) = e$ if $x \in \mathcal{E}_2$ and $x - e \in V_0(e)$, using the operations in $\mathcal{H}_3(\mathbb{O})$.

3.2.3. *The boundary of the exceptional domain of type V.* Along the same lines, we describe the manifolds of tripotents of the exceptional Jordan triple system of type V and the boundary of the associated symmetric domain.

PROPOSITION 3.6. *Let $W = V_1(e_1) = \mathcal{F}_2 \oplus \mathcal{F}_3 \subset H_3(\mathbb{O})$ be the exceptional Jordan triple system of type V . Then*

1. *The manifold \mathcal{E}'_1 of minimal tripotents is defined by*

$$\begin{aligned} \mathcal{E}'_1 &= \{e \in W \mid \{e, e, e\} = 2e, (e \mid e) = 1\} \\ &= \{e \in W \mid e^\# = 0, (e \mid e) = 1\}. \end{aligned}$$

For $e \in \mathcal{E}'_1$, we have $W_2(e) = \mathbb{C}e$, $\dim_{\mathbb{C}} W_1(e) = 10$. The manifold \mathcal{E}'_1 has real dimension 21 and is a Cauchy-Riemann manifold of CR type $(10, 1)$.

2. *The manifold \mathcal{E}'_2 of maximal tripotents is defined by*

$$\begin{aligned} \mathcal{E}'_2 &= \{e \in W \mid \{e, e, e\} = 2e, (e \mid e) = 2\} \\ &= \{e \in W \mid (e^\# \mid e^\#) = 1, (e \mid e) = 2\}. \end{aligned}$$

For $e \in \mathcal{E}'_2$, we have $\dim W_2(e) = 8$ and $\dim W_1(e) = 8$. The manifold \mathcal{E}'_2 has real dimension 24 and is a Cauchy-Riemann manifold of CR type $(8, 8)$.

PROOF. The description of \mathcal{E}'_1 and \mathcal{E}'_2 has been given in Proposition 2.19.

1. From Lemma 2.17, we can take as minimal tripotent $u = F_2(\beta)$, where $\beta \in \mathbb{O}$ such that $(\beta \mid \beta) = 1$ and $n(\beta) = 0$. The spaces of the Peirce decomposition of W with respect to u are

$$W_0(u) = \mathbb{C}\overline{u} \oplus F_3(\ker L(\overline{\beta})),$$

$$\begin{aligned} W_1(u) &= F_2(\langle \beta, \bar{\beta} \rangle^\perp) \oplus F_3(\ker L(\beta)), \\ W_2(u) &= \mathbb{C}u. \end{aligned}$$

From Lemma 2.16, we know that

$$\dim_{\mathbb{C}} \ker L(\beta) = \dim_{\mathbb{C}} \ker L(\bar{\beta}) = 4,$$

which yields $\dim W_0(u) = 5$ and $\dim W_1(u) = 10$.

2. We can choose as maximal tripotent $w = u + v = F_2(\beta + \bar{\beta}) = F_2(c)$, with $c \in \mathbb{O}_c$ and $n(c) = 1$. The Peirce spaces for w are $W_2(w) = \mathcal{F}_2$, $W_1(w) = \mathcal{F}_3$; both have complex dimension 8. \square

PROPOSITION 3.7. *The boundary $\partial\Omega'$ of the exceptional symmetric domain $\Omega' = \Omega_W$ of type V is the disjoint union*

$$\partial\Omega' = \partial_1\Omega' \amalg \partial_2\Omega' \quad (3.10)$$

of the locally closed manifold $\partial_1\Omega'$ and of the compact manifold $\partial_2\Omega'$.

1. *The smooth boundary part $\partial_1\Omega'$ is a locally closed submanifold of real codimension 1; it contains the manifold*

$$\mathcal{E}'_1 = \{x \in W; (x | x) = 1, x^\# = 0\}.$$

of minimal tripotents of $W = \mathcal{F}_2 \oplus \mathcal{F}_3$. For $e \in \mathcal{E}'_1$, the normal direction at e to $\partial_1\Omega'$ is e ; the direction $\overrightarrow{T_e(\partial_1\Omega')}$ of the tangent space $T_e(\partial_1\Omega')$ is

$$\overrightarrow{T_e(\partial_1\Omega')} = i\mathbb{R}e \oplus W_1(e) \oplus W_0(e),$$

where the Peirce subspaces $W_1(e)$ and $W_0(e)$ have respective complex dimensions 10 and 5. The affine component $e + \Omega'(e)$ is the unit Hermitian ball with center e in $e + W_0(e)$.

2. *The submanifold $\partial_2\Omega' = \mathcal{E}'_2$ is a compact, Cauchy-Riemann submanifold of CR type (16, 8) and real dimension 24.*

PROOF. 1. Consider the minimal tripotent $u = F_2(\beta)$, where $\beta \in \mathbb{O}$ such that $(\beta | \beta) = 1$ and $n(\beta) = 0$. From Lemma 2.17, we know that the spaces of the Peirce decomposition of W with respect to u are

$$\begin{aligned} W_0(u) &= \mathbb{C}\bar{u} \oplus F_3(\ker L(\bar{\beta})), \\ W_1(u) &= F_2(\langle \beta, \bar{\beta} \rangle^\perp) \oplus F_3(\ker L(\beta)), \\ W_2(u) &= \mathbb{C}u. \end{aligned}$$

This proves the statement about the tangent space and the dimensions of Peirce subspaces.

Let v be a tripotent in $W_0(u)$. Then v is orthogonal to u and (u, v) is a frame of W . This implies that v is maximal in $W_0(u)$ and that the JTS $W_0(u)$ is of rank 1, so that $\Omega(u)$ is a Hermitian ball in $W_0(u)$.

2. An element $x \in W$ belongs to $\partial_2\Omega'$ if and only if

$$\begin{aligned} g_1(x, x) &\equiv 1 - (x | x) + (x^\# | x^\#) = 0, \\ g_2(x, x) &\equiv 2 - (x | x) = 0. \end{aligned}$$

These conditions are clearly equivalent to

$$(x | x) = 2, \quad (x^\# | x^\#) = 1,$$

which is precisely the characterization of elements of \mathcal{E}'_2 . So $\partial_2\Omega' = \mathcal{E}'_2$, which also results from the general theory. The structure of \mathcal{E}'_1 has been given in Proposition 3.6. \square

EXERCISE 3.4. 1. For a minimal tripotent $e = F_2(\beta) + F_3(\gamma) \in \mathcal{E}'_1$, compute explicitly the Peirce subspaces $W_1(e)$ and $W_0(e)$ and their Jordan triple structure.
2. Compute explicitly the map

$$p'_1 : \partial_1\Omega' \rightarrow \mathcal{E}'_1$$

defined by $p'_1(x) = e$ if $x \in \mathcal{E}'_1$ and $x - e \in W_0(e)$.

3. Identify the type of the Hermitian positive JTS $W_1(e)$, where e is a minimal tripotent of W .

EXERCISE 3.5. For a maximal tripotent $e = F_2(\beta) + F_3(\gamma) \in \mathcal{E}'_2$, compute explicitly the Peirce subspaces $W_2(e)$ and $W_1(e)$ and their Jordan triple structure.

3.3. Compactification of exceptional symmetric domains. In this section, we work out the canonical projective realization of the compact dual of the two exceptional domains (see [5], [7]).

3.3.1. *The Freudenthal manifold.* Consider the exceptional Jordan triple $V = H_3(\mathbb{O})$ of type VI. The *generic norm* of V is

$$N_V(x, y) = m_V(1, x, y) = 1 - (x \mid y) + (x^\# \mid y^\#) - \det x \det \bar{y}.$$

To this generic norm, we associate the map

$$\begin{aligned} j : V &\rightarrow \mathbb{P}(\mathbb{C} \oplus V \oplus V \oplus \mathbb{C}) \\ x &\mapsto [1, x, x^\#, \det x], \end{aligned}$$

where $[\dots]$ denotes the class in the projective space.

DEFINITION 8. *The Freudenthal manifold is the submanifold of $\mathbb{P}(\mathbb{C} \oplus V \oplus V \oplus \mathbb{C})$ defined by*

$$\mathcal{M} = \{[\lambda, x, y, \mu] \mid \lambda, \mu \in \mathbb{C}, x, y \in V, y^\# = \mu x, x^\# = \lambda y, (x : y) = 3\lambda\mu\}.$$

Note that this definition makes sense, for the defining equations $y^\# = \mu x$, $x^\# = \lambda y$ and $(x : y) = 3\lambda\mu$ are homogeneous of degree 2. As $(x^\#)^\# = x \det x$ and $(x : x^\#) = 3 \det x$ for $x \in V$, we have $j(V) \subset \mathcal{M}$. The map j is clearly an immersion.

Let $[\lambda, x, y, \mu] \in \mathcal{M}$ and assume $\lambda \neq 0$. Let $x' = \frac{x}{\lambda}$; then

$$\begin{aligned} (x')^\# &= \frac{x^\#}{\lambda^2} = \frac{y}{\lambda}, \\ \det(x') &= \frac{\det x}{\lambda^3} = \frac{1}{3} \frac{(x^\# : x)}{\lambda^3} = \frac{1}{3} \frac{(y : x)}{\lambda^2} = \frac{\mu}{\lambda}, \end{aligned}$$

which shows that $[\lambda, x, y, \mu] = j(x')$.

PROPOSITION 3.8. *The map j is an immersion of V onto an open dense subset of the Freudenthal manifold \mathcal{M} .*

3.3.2. *Compactification of the 16-dimensional exceptional domain.* Consider the exceptional symmetric domain of dimension 16, realized as

$$W = \mathcal{F}_2 \oplus \mathcal{F}_3 \subset V = H_3(\mathbb{O}).$$

The *generic norm* of W is

$$N_W(x, y) = m_W(1, x, y) = 1 - (x \mid y) + (x^\# \mid y^\#).$$

One checks easily from the definition of $x^\#$ that $x \in W$ implies $x^\# \in V_0(e_1)$. Note that the Peirce decomposition of V with respect to e_1 has the eigenspaces

$$V_2(e_1) = \mathbb{C}e_1, \quad V_1(e_1) = W = \mathcal{F}_2 \oplus \mathcal{F}_3,$$

$$V_0(e_1) = \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathcal{F}_1.$$

LEMMA 3.9. *Let $z = e_1 + x + y$ with $x \in W$ and $y \in V_0(e_1)$. Then $z^\# = 0$ if and only if $y = -e_1 \times x^\#$.*

PROOF. Let $z = e_1 + x + y$, $x = F_2(b) + F_3(c) \in W$, $y = \mu e_2 + \nu e_3 + F_1(a) \in V_0(e_1)$. We have

$$e_1^\# = 0, \quad x^\# = -n(b)e_2 - n(c)e_3 + \widetilde{F}_1(bc),$$

$$y^\# = (\mu\nu - n(a))e_1, \quad e_1 \times x = 0,$$

$$e_1 \times y = \mu e_3 + \nu e_2 - F_1(a),$$

$$x \times y = -\mu F_2(b) - \nu F_3(c) + \widetilde{F}_3(ab) + \widetilde{F}_2(ca)$$

and

$$\begin{aligned} z^\# &= (e_1 + x + y)^\# = e_1^\# + x^\# + y^\# + e_1 \times x + e_1 \times y + x \times y \\ &= -n(b)e_2 - n(c)e_3 + \widetilde{F}_1(bc) + (\mu\nu - n(a))e_1 \\ &\quad + \mu e_3 + \nu e_2 - F_1(a) - \mu F_2(b) - \nu F_3(c) + \widetilde{F}_3(ab) + \widetilde{F}_2(ca) \\ &= (\mu\nu - n(a))e_1 + (\nu - n(b))e_2 + (\mu - n(c))e_3 \\ &\quad + \widetilde{F}_1(bc) - F_1(a) + \widetilde{F}_2(ca) - \mu F_2(b) + \widetilde{F}_3(ab) - \nu F_3(c). \end{aligned}$$

Then $z^\# = 0$ implies

$$\mu = n(c), \quad \nu = n(b), \quad a = \widetilde{bc}, \tag{3.11}$$

that is,

$$y = n(c)e_2 + n(b)e_3 + \widetilde{F}_1(bc),$$

which is equivalent to

$$y = -e_1 \times x^\#.$$

Conversely, if $y = -e_1 \times x^\#$, the relations (3.11) are satisfied and imply

$$n(a) = n(bc) = n(b)n(c) = \mu\nu,$$

$$\widetilde{ca} = (bc)\widetilde{c} = \mu b, \quad \widetilde{ab} = \widetilde{b}(bc) = \nu c,$$

which shows that $z^\# = 0$. \square

With the help of the previous lemma, we are now able to describe a compactification of W , isomorphic to the canonical compactification associated to the generic norm N_W .

PROPOSITION 3.10. *Let $V = H_3(\mathbb{O})$ and $W = \mathcal{F}_2 \oplus \mathcal{F}_3$. Define $j : W \rightarrow \mathbb{P}(V)$ by*

$$j(x) = [e_1 + x - e_1 \times x^\#] \quad (x \in W).$$

Then j is a biholomorphism of W onto an open dense subset of the manifold

$$\mathcal{P} = \{[z] \in \mathbb{P}(V) \mid z^\# = 0\}.$$

Indeed, j is an immersion and maps W biholomorphically onto

$$j(W) = \{[z] \in \mathcal{P} \mid (z : e_1) \neq 0\}.$$

The manifold \mathcal{P} is the image in $\mathbb{P}(V)$ of the cone $\{z^\# = 0\}$ of rank one elements in V .

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